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Extension theory of operators in Krein and Pontryagin spaces and applications

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	Sivumäärä 109	**Kieli** Englanti

Julkaisun nimike

Krein ja Pontryagin -avaruuksien operaattorien laajennukset ja niiden sovelluksia

Tiivistelmä

Väitöskirjassa tarkastellaan operaattoreiden laajentamiseen liittyvää ongelma- ja sovelluskenttää. Ensimmäisenä uutena tuloksena työssä yleistetään eräs Yu.L. Shmul'yanin todistama teoreema, joka koskee epätäydellisen lohko-operaattorin täydentämistä ei-negatiiviseksi operaattoriksi. Alkuperäinen ei-negatiivisuusehto korvataan onnistuneesti huomattavasti heikommilla oletuksilla, jotka liittyvät operaattoreiden negatiivisiin spektreihin. Tulokset antavat yleiset ratkeavuusehdot esitetyille täydennysongelmille sekä myös täydellisen kuvauksen kaikista ongelman ratkaisuista. Nämä keskeiset tulokset yleistetään myös Krein -avaruuksien operaattoreille ja niitä sovelletaan useisiin erilaisiin ongelmiin, jotka voidaan palauttaa operaattoreiden laajennuksien tarkasteluun Hilbert, Pontryagin ja Krein -avaruuksissa. Eräänä seurauksena yleistetään muun muassa M.G. Kreinin kuuluisa tulos, joka karakterisoi symmetrinen kontraktiivisen kuvauksen kaikki itseadjungoidut kontraktiiviset operaattorilaajennukset, tilanteeseen, jossa operaattorit ovat kvasikontraktiivisia Hilbert, Pontryagin tai Krein -avaruuksissa. Lisäksi työssä tarkastellaan J-kontraktiivisten operaattorien nosto-ongelmia mainituissa avaruuksissa kuten myös todistetaan yleistyksiä Hilbert ja Pontryagin -avaruuksien operaattoreiden faktorointiin liittyen.

Toisena laajennusteoriaan liittyvänä tutkimusalueena väitöskirjassa ovat reunakolmikot ja niihin liittyvät Weyl-funktiot. Tässä työssä nämä käsitteet määritellään Pontryagin-avaruuden isometriselle kuvaukselle V ja työssä johdetaan niiden keskeiset ominaisuudet. Esitetyn reunakolmikkoja koskevan teorian sovelluksina johdetaan muun muassa kaava, joka kuvaa isometrisen operaattorin V kaikki yleistetyt resolventit. Teorian avulla johdetaan myös tunnetulle Arov-Grossmanin kaavalle vastine Pontryagin-avaruuksien isometrisen kuvauksen V tapauksessa; kaava kuvaa operaattorin V kaikkiin unitaarisiin laajennuksiin liittyvät sirontamatriisit.

Asiasanat

Täydennys, operaattorilaajennukset, Hilbert-avaruus, Pontryagin-avaruus, Krein-avaruus, symmetrinen operaattori, itseadjungoitu laajennus, isometrinen operaattori, unitaarinen laajennus, reunakolmikko, Weyl-funktio, yleistetty resolventti, sirontamatriisi.

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Abstract

In this dissertation various types of operator extension problems are investigated. A first new result is a generalization of a theorem due to Yu.L. Shmul'yan on completion of nonnegative block operators. The initial nonnegativity condition is relaxed and replaced with much weaker conditions on the negative part of spectra. Some general solvability criteria and descriptions of all solutions for analogous completion problems are obtained. Such results are also presented for operators acting in Krein spaces and applied to various problems which can be treated using the machinery of extension theory of operators in Hilbert, Pontryagin, and Krein spaces. For instance, a famous result of M.G. Krein concerning the description of selfadjoint contractive extensions of a Hermitian contraction is extended to the case of quasi-contractions in Hilbert, Pontryagin, and Krein spaces. Furthermore, some lifting problems for J-contractive operators in Hilbert, Pontryagin, and Krein spaces are treated and some generalizations concerning factorization of Hilbert and Pontryagin space operators are derived.

A related area of research concerns boundary triplets and Weyl functions. These concepts are defined for an isometric operator V acting on a Pontryagin space and their basic properties are established. After the boundary triplet technique in this setting is developed the results are applied to derive a formula that describes all generalized resolvents of an isometric operator V. In the setting of scattering matrices of unitary extensions of V this technique is used to prove a Pontryagin space version of the Arov-Grossman formula, which describes all scattering matrices of unitary extensions of the isometric operator V.

Keywords

Completion, extensions of operators, Hilbert space, Pontryagin space, Krein space, symmetric operator, selfadjoint extension, isometric operator, unitary extension, boundary triplet, Weyl function, generalized resolvent, scattering matrix.

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I'm happy that I got a possibility to continue my Ph.D. research at the University of Vaasa. This dissertation got its present form under supervision of Prof. Seppo Hassi. We have spent an uncountable number of hours in discussions and I admire his deep knowledge in Mathematics. My deep and sincere gratitude to him is hard to put in words. I also would like to extend my gratitude to all the members of the Department of Mathematics and Statistics of the University of Vaasa for the friendly working environment. Furthermore, I am thankful to the University of Vaasa for their financial support which made it possible to attend a number of conferences and workshops.

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Vaasa, June 2016
Dmytro Baidiuk

CONTENTS

LIST OF PUBLICATIONS

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(II) Baidiuk, D. (2016). Completion and extension of operators in Kreĭn spaces. Submitted. Preprint version published in *Proceedings of the University of Vaasa*. Working Papers 10, Mathematics 6.

(III) Baidiuk, D. (2013). On boundary triplets and generalized resolvents of an isometric operators in a Pontryagin space. *Journal of Mathematical Sciences* 194, 5, 513–531.

(IV) Baidiuk, D. (2013). Description of scattering matrices of unitary extensions of isometric operators in a Pontryagin space. *Mathematical Notes* 94, 6, 940–943.

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AUTHOR'S CONTRIBUTION

Publication I: "Completion, extension, factorization, and lifting of operators"

This article represents a joint discussion and all the results are a joint work with Seppo Hassi.

Publication II: "Completion and extension of operators in Kreĭn spaces"

This is an independent work of the author.

Publication III: "On boundary triplets and generalized resolvents of an isometric operators in a Pontryagin space"

This is an independent work of the author. The topic was proposed by Volodymyr Derkach.

Publication IV: "Description of scattering matrices of unitary extensions of isometric operators in a Pontryagin space"

This is an independent work of the author.

1 INTRODUCTION

The theory of extensions of symmetric and isometric operators in Hilbert spaces was initiated by J. von Neumann in the early 1930s. This theory has numerous applications to different problems of mathematical physics and analysis, in particular perturbation theory of operators as well as classical problems of analysis like the moment problem. The literature devoted to such applications is very extensive (see Kreĭn (1946), Albeverio & Kurasov (1999), Reed & Simon (1975), Pavlov (1987), Kostenko & Malamud (2010) and references therein).

In the paper of Kreĭn (1947) it was proved that for a densely defined nonnegative operator A in a Hilbert space there are two extremal extensions of A, the Friedrichs (hard) extension A_F and the Kreĭn-von Neumann (soft) extension A_K, such that every nonnegative selfadjoint extension $\widetilde{A}$ of A can be characterized by the following two inequalities:

$$(A_F + a)^{-1} \leq (\widetilde{A} + a)^{-1} \leq (A_K + a)^{-1}, \quad a > 0.$$

Later the study of nonnegative selfadjoint extensions of $A \geq 0$ was generalized to the case of nondensely defined operators $A \geq 0$ by Ando & Nishio (1970), as well as to the case of linear relations (multivalued linear operators) $A \geq 0$ by Coddington & de Snoo (1978). The extension theory of unbounded symmetric Hilbert space operators and related resolvent formulas originating also from Kreĭn (1944, 1946), see also e.g. Langer & Textorius (1977), was generalized to the spaces with indefinite inner products in the well-known series of papers by H. Langer and M.G. Kreĭn, see e.g. Kreĭn & Langer (1971), and all of this has been further investigated, developed, and extensively applied in various other areas of mathematics and physics by numerous other researchers.

An other approach to the investigation of selfadjoint extensions of symmetric operators is based on the notion of boundary triplets. In many cases, the boundary triplets' method has appeared to offer a more convenient tool than the classical methods of extension theory, for instance, when treating various spectral and scattering properties of differential operators. In fact, initially this method was systematically studied and elaborated by J.W. Calkin in his 1937 Harvard doctoral dissertation and then published in the paper Calkin (1939), as a generalization of the method of boundary conditions used in the theory of Sturm-Liouville problems to the case of arbitrary symmetric operators. However, the method was not widespread at that time, apparently, because of the complexity of the language and the abstract nature of the work. Later on the boundary value space technique has been extensively developed in the works of Ukrainian mathematicians (F. Rofe-Beketov, M. Gorbachuk, V. Lyantse, A. Kochubei, O. Storozh, M. Malamud, V. Derkach, and others, see Albeverio & Kurasov (1999); Gorbachuk & Gorbachuk (1990); Derkach & Malamud (1991); Malamud (1992) and the bibliography therein).

Another important concept in the boundary triplets' theory is the so-called Weyl function of a symmetric operator, which is a natural generalization of the classical

Weyl-Titchmarsh m-function appearing in the Sturm-Liouville theory. The definition of abstract Weyl functions associated with boundary triplets was proposed by V. Derkach and M. Malamud. In a series of works (see Derkach & Malamud (1991); Malamud (1992) and references therein), these authors investigated properties of the Weyl function and applied them, for instance, to the spectral analysis of selfadjoint extensions of symmetric operators. More recently, the theory of boundary triplet has been further developed in the serious of papers (see Derkach et. al. (2006), Derkach et. al. (2009), and Derkach et. al. (2012)) where so-called Nevanlinna families are appearing as the associated Weyl functions.

It is well known that the extension theory of symmetric operators can be successfully applied not only to boundary value problems and singularly perturbed operators, but also to various classical problems like moment problems and Nevanlinna–Pick type interpolation problems. The main role in this approach to such classical problems is played by the Kreĭn's formula for generalized resolvents of a symmetric operator A in a Hilbert space $\mathcal{H}$ (see Kreĭn (1946)). Another proof of this formula which is based on the notion of the boundary relation and the coupling has been developed in Derkach et. al. (2009). Closely related to the notion of generalized resolvents is the concept of $\mathfrak{L}$-resolvent for a subspace $\mathfrak{L}$ of $\mathcal{H}$, the compressed resolvent $P_{\mathfrak{L}}(\widetilde{A}-\lambda)^{-1} \upharpoonright \mathfrak{L}$ either of an exit space or a canonical selfadjoint extension $\widetilde{A}$ of A is called the $\mathfrak{L}$-resolvent of A. The set of all $\mathfrak{L}$-resolvents of A was described in Kreĭn (1946) via the formula

$$P_{\mathfrak{L}}(\widetilde{A}-\lambda)^{-1} \upharpoonright \mathfrak{L} = \big(W_{11}(\lambda)\tau(\lambda) + W_{12}(\lambda)\big)\big(W_{21}(\lambda)\tau(\lambda) + W_{22}(\lambda)\big)^{-1}$$

where $W_{A,\mathfrak{L}}(\lambda) = \big(W_{ij}(\lambda)\big)_{i,j=1}^{2}$ is the so-called $\mathfrak{L}$ - resolvent matrix of A, and the parameter τ ranges over the class $\widetilde{R}_{\mathfrak{L}}$ of Nevanlinna families with values in $\mathcal{B}(\mathfrak{L})$. From the above resolvent formula one obtains also a description of the set of all $\mathfrak{L}$ - spectral functions $P_{\mathfrak{L}}E_{\widetilde{A}}(\cdot) \upharpoonright \mathfrak{L}$ by means of Cauchy's formula and in applications to classical problem, like the Hamburger moment problem, this leads to a description of all the solutions. The theory of $\mathfrak{L}$-resolvent matrices of an operator A has been developed by M.G. Kreĭn and Sh. Saakyan (Kreĭn (1946), Kreĭn & Saakyan (1966)). Further developments as well as their connections with the theory of boundary triples and characteristic functions of nonselfadjoint operators can be found in Derkach & Malamud (1991, 1995).

A description of generalized resolvents of a standard symmetric operator in a Pontryagin space (i.e. with nondegenerate defect subspaces) was obtained in Kreĭn & Langer (1971) and in Dijksma et. al. (1990). The notions of a boundary triplet and the corresponding Weyl function were generalized to the case of a symmetric operator in a Pontryagin space by Derkach (1995). The theory of $\mathfrak{L}$-resolvent matrices of a symmetric operator in a Pontryagin space setting was developed in Derkach (1999).

Extension theory of isometric operators have been applied to interpolation problems in Schur classes by V. Adamjan, D. Arov and M.G. Kreĭn in Adamjan et al. (1968), Arov (1993). In this case the main role is played by the description of scattering

matrices of unitary extensions of an isometric operator. Such a description was obtained by Arov & Grossman (1992) as a parallel version of the M.G. Kreĭn's formula for generalized resolvents of symmetric operators.
Malamud & Mogilevskii (2003) developed the theory of boundary triplets for isometric operators. They introduced the notion of the Weyl function of an isometric operator and applied it to the theory of generalized resolvents of isometric operators.Then in Malamud & Mogilevskii (2005) the theory of $\mathfrak{L}$-resolvent matrices of an isometric operator was elaborated.
As was indicated in Adamjan et. al. (1971) the Nehari-Takagi problem can be reduced to an extension problem for an isometric operator in a Pontryagin space. A description of generalized coresolvents and $\mathfrak{L}$-resolvents of a standard isometric operator (whose domain is a nondegenerate subspace) was obtained in Langer (1971), Langer & Sorjonen (1974) and in Dijksma et. al. (1990). For a nonstandard isometric operator a description of generalized coresolvents was obtained in Nitz (2000a), Nitz (2000b). This description turned out to be quite complicated, since such an operator admits multivalued unitary extensions (unitary relations).

2 KREIN AND PONTRYAGIN SPACES

2.1 Definitions and general facts

We start with some basic definitions related to Kreĭn spaces (see Azizov & Iokhvidov (1989) and Bognar (1974)).

A Kreĭn space $\mathcal{H}$ is a topological complex linear space $\mathcal{H}$ equipped with a scalar product $[\cdot,\cdot]$, such that for some continuous linear operator J in the space $\mathcal{H}$ with the property $J^2 = J$, the new scalar product $(\cdot,\cdot)_J := [J\cdot,\cdot]$ turns $\mathcal{H}$ into a Hilbert space. It follows, that the topology of the Hilbert space is equivalent to the topology of the Kreĭn space. The operator J is called a fundamental symmetry or a signature operator. It is easy to see, that J is selfadjoint with respect to both scalar products. A Kreĭn space with fundamental symmetry J is denoted by $(\mathcal{H}, J)$.

A vector $h \in \mathcal{H}$ is called positive, neutral or negative if $[h,h] > 0$, $[h,h] = 0$ or $[h,h] < 0$, respectively. A subspace of a Kreĭn space $\mathfrak{L}$ is called positive, neutral or negative if every nonzero vector $h \in \mathfrak{L}$ is positive, neutral or negative, respectively. Below some standard notations are given:

$$
\begin{aligned}
x[\perp]y \ &:\Leftrightarrow [x,y] = 0;\\
\mathfrak{L}_1 \dot{+} \mathfrak{L}_2 &:= \mathfrak{L}_1 + \mathfrak{L}_2 \text{ if } \mathfrak{L}_1 \cap \mathfrak{L}_2 = 0;\\
\mathfrak{L}^{[\perp]} &:= \{x \in \mathcal{H} : \text{ for all } y \in \mathfrak{L},\ x[\perp]y\};\\
\mathfrak{L}_1[\dot{+}]\mathfrak{L}_2 &:= \mathfrak{L}_1 + \mathfrak{L}_2 \text{ if } \mathfrak{L}_1 \cap \mathfrak{L}_2 = 0, \mathfrak{L}_1[\perp]\mathfrak{L}_2;\\
\mathfrak{L}_1[-]\mathfrak{L}_2 &:= \mathfrak{L}_2 \cap \mathfrak{L}_1^{[\perp]} \text{ if } \mathfrak{L}_1 \subset \mathfrak{L}_2.
\end{aligned}
$$

A regular subspace of a Kreĭn space means a closed subspace $\mathfrak{L} \subset \mathcal{H}$ which is a Kreĭn space in the scalar product of $\mathcal{H}$. A subspace $\mathfrak{L} \subset \mathcal{H}$ is regular if and only if $\mathfrak{L}[+]\mathfrak{L}^{[\perp]} = \mathcal{H}$.

Every closed subspace $\mathfrak{L}$ of a Kreĭn space $\mathcal{H}$ admits a decomposition of the form

$$\mathfrak{L} = \mathfrak{L}_+[\dot{+}]\mathfrak{L}_-[\dot{+}]\mathfrak{L}_0,$$

where $\mathfrak{L}_+, \mathfrak{L}_-, \mathfrak{L}_0$ are positive, negative, and neutral closed subspaces, respectively. The subspace $\mathfrak{L}_0$ is uniquely defined and can be found by the formula $\mathfrak{L}_0 = \mathfrak{L} \cap \mathfrak{L}^{[\perp]}$. It is called the isotropic part of $\mathfrak{L}$. In general, the subspaces $\mathfrak{L}_\pm$ are not unique but their dimensions do not depend on the choice and are called the signature indices of $\mathfrak{L}$, and are denoted by $\kappa_0(\mathfrak{L}) = \dim \mathfrak{L}_0$, $\kappa_\pm(\mathfrak{L}) = \dim \mathfrak{L}_\pm$. The whole Kreĭn space $\mathcal{H}$ has no isotropic part, i.e. $\kappa_0(\mathcal{H}) = 0$, so it has a decomposition $\mathcal{H} = \mathcal{H}_+[\dot{+}]\mathcal{H}_-$, which is called a fundamental decomposition. The number $\kappa_-(\mathcal{H})$ is often called the number of negative squares of a Kreĭn space $\mathcal{H}$. A Kreĭn space $\mathcal{H}$ is called a Pontryagin space if $\kappa_-(\mathcal{H}) < \infty$. If $(\mathcal{H}, J)$ is a Kreĭn space then $\kappa_\pm(\mathcal{H}) = \nu_\pm(J)$; the negative and the positive indices of inertia of J.

2.2 Linear relations in Pontryagin spaces

We denote by $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ the set of all continuous and everywhere defined linear operators from the Pontryagin space $\mathcal{H}_1$ to the Pontryagin space $\mathcal{H}_2$; we write $\mathcal{B}(\mathcal{H})$ instead of $\mathcal{B}(\mathcal{H}, \mathcal{H})$. The graph of a linear operator $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ is a closed subspace of $\mathcal{H}_1 \times \mathcal{H}_2$, defined by

$$\operatorname{gr} T = \left\{ \begin{bmatrix} x \\ Tx \end{bmatrix} : x \in \mathcal{H}_1 \right\}.$$

A linear relation (l.r.) T from $\mathcal{H}_1$ to $\mathcal{H}_2$ is a linear subspace in $\mathcal{H}_1 \times \mathcal{H}_2$. If the linear operator T is identified with its graph, then the set $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ of linear bounded operators from $\mathcal{H}_1$ to $\mathcal{H}_2$ is contained in the set of linear relations from $\mathcal{H}_1$ to $\mathcal{H}_2$. In what follows, we interpret the linear relation $T : \mathcal{H}_1 \to \mathcal{H}_2$ as a multivalued linear mapping from $\mathcal{H}_1$ to $\mathcal{H}_2$. If $\mathcal{H} := \mathcal{H}_1 = \mathcal{H}_2$ we say that T is a linear relation in $\mathcal{H}$.
For the linear relation $T : \mathcal{H}_1 \to \mathcal{H}_2$, we denote by $\operatorname{dom} T$, $\ker T$, $\operatorname{ran} T$, and $\operatorname{mul} T$ the domain, the kernel, the range, and the multivalued part of T, respectively. The inverse relation T^{-1} is a linear relation from $\mathcal{H}_2$ to $\mathcal{H}_1$ defined by the equality

$$T^{-1} = \left\{ \begin{bmatrix} f' \\ f \end{bmatrix} : \begin{bmatrix} f \\ f' \end{bmatrix} \in T \right\}.$$

The operator sum $T + S$ of two linear relations T and S is defined by

$$T + S = \left\{ \begin{bmatrix} f \\ g + h \end{bmatrix} : \begin{bmatrix} f \\ g \end{bmatrix} \in T, \begin{bmatrix} f \\ h \end{bmatrix} \in S \right\}.$$

Consider two Pontryagin spaces $(\mathcal{H}_1, j_{\mathcal{H}_1})$ and $(\mathcal{H}_2, j_{\mathcal{H}_2})$ and a linear relation T from $\mathcal{H}_1$ to $\mathcal{H}_2$. Then the adjoint linear relation $T^{[*]}$ consists of pairs $\begin{bmatrix} g_2 \\ g_1 \end{bmatrix} \in \mathcal{H}_2 \times \mathcal{H}_1$ such that

$$[f_2, g_2]_{\mathcal{H}_2} = [f_1, g_1]_{\mathcal{H}_1}, \text{ for all } \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \in T.$$

If T^* is the l.r. adjoint to T considered as a l.r. from the Hilbert space $\mathcal{H}_1$ to the Hilbert space $\mathcal{H}_2$, then $T^{[*]} = j_{\mathcal{H}_1} T^* j_{\mathcal{H}_2}$.

Definition 2.1. A linear relation T from a Pontryagin space $(\mathcal{H}_1, j_{\mathcal{H}_1})$ to a Pontryagin space $(\mathcal{H}_2, j_{\mathcal{H}_2})$ is called isometric, if for all $\begin{bmatrix} \varphi \\ \varphi' \end{bmatrix} \in T$ the equality

$$[\varphi', \varphi']_{\mathcal{H}_2} = [\varphi, \varphi]_{\mathcal{H}_1} \tag{2.1}$$

holds. Moreover, T is called contractive (expansive), if equality (2.1) is replaced by an inequality with the sign $\leq$ (by $\geq$, respectively). It follows from (2.1) that a linear relation T is isometric if and only if $T^{-1} \subset T^{[*]}$. A linear relation from $(\mathcal{H}_1, j_{\mathcal{H}_1})$

to $(\mathcal{H}_2, j_{\mathcal{H}_2})$ is called unitary, if $T^{-1} = T^{[*]}$.

3 UNITARY COLLIGATIONS, SCATTERING MATRICES, AND GENERALIZED RESOLVENTS

We recall the basic notions of the theory of unitary colligations (see Alpay et. al. (1997), Brodskii (1978)). Let $\mathcal{H}$ be a Pontryagin space with a negative index κ, let $\mathfrak{N}_2$ and $\mathfrak{N}_1$ be Hilbert spaces, and let $U = \begin{pmatrix} T & F \\ G & H \end{pmatrix}$ be a unitary operator from $\mathcal{H} \oplus \mathfrak{N}_2$ to $\mathcal{H} \oplus \mathfrak{N}_1$. Then the quadruple $\Delta = (\mathcal{H}, \mathfrak{N}_2, \mathfrak{N}_1; U)$ is called a unitary colligation. The spaces $\mathcal{H}, \mathfrak{N}_2, \mathfrak{N}_1$ are called, respectively, the state space, the input channel space, and the output channel space, and the operator U is called the connecting operator of the colligation Δ.
The colligation Δ is called simple, if there exists no subspace in the space $\mathcal{H}$ reducing U. The operator function

$$\Theta(\lambda) = H + \lambda G(I - \lambda T)^{-1}F : \mathfrak{N}_2 \to \mathfrak{N}_1 \quad (\lambda^{-1} \in \rho(T))$$

is called the *characteristic function* of a colligation Δ or the *scattering matrix* of the unitary operator U relative to the channel spaces $\mathfrak{N}_2$ and $\mathfrak{N}_1$ in the case where $\mathfrak{N}_2, \mathfrak{N}_1, \mathcal{H}$ are Hilbert spaces; see Arov & Grossman (1992). The characteristic function characterizes a simple unitary colligation up to unitary equivalence. The characteristic function can be also expressed as follows.

Proposition 3.1. *(Derkach (2001)) Let $\Delta = (\mathcal{H}, \mathfrak{N}_2, \mathfrak{N}_1; T, F, G, H)$ be a unitary colligation and $\Theta(\cdot)$ be the characteristic function of this colligation. Then*

$$\Theta(\lambda) = P_{\mathfrak{N}_1}(I - \lambda U P_{\mathcal{H}})^{-1}U \restriction \mathfrak{N}_2 = P_{\mathfrak{N}_1}U(I - \lambda P_{\mathcal{H}}U)^{-1} \restriction \mathfrak{N}_2,$$

where $P_{\mathcal{H}}$ and $P_{\mathfrak{N}_i}$ are orthoprojections from $\mathcal{H} \oplus \mathfrak{N}_i$ onto $\mathcal{H}$ and $\mathfrak{N}_i$ $(i = 1, 2)$, respectively.

In the sequel, we need the Schur class S and the generalized Schur class S_κ of functions. The definition reads as follows (see Alpay et. al. (1997)).

Definition 3.2. A function $s(\lambda)$ defined and holomorphic in a domain $\mathfrak{h}_s \subset \mathbb{D}$ belongs to the class $S_\kappa(\mathfrak{N}_1, \mathfrak{N}_2)$, if the kernel

$$K_\mu(\lambda) = \frac{1 - s(\mu)^* s(\lambda)}{1 - \lambda\overline{\mu}}$$

has κ negative squares, i.e. for all $\lambda_1, ..., \lambda_n \in \mathfrak{h}_s$ and $u_1, ..., u_n \in \mathfrak{N}_1$ the matrix $((K_{\lambda_j}(\lambda_i)u_i, u_j))_{i,j=1}^n$ has at most κ negative eigenvalues and at least for one such choice it has exactly κ negative eigenvalues.

In particular, an $[\mathfrak{N}_1, \mathfrak{N}_2]$-valued function $s(\cdot)$ belongs to the class $S(\mathfrak{N}_1, \mathfrak{N}_2)$, if the kernel $K_\mu(\lambda)$ is positive definite everywhere in $\mathbb{D}$. As is known, the last condition is equivalent to $s(\cdot)$ being holomorphic in $\mathbb{D}$ and $\|s(\lambda)\| \le 1$ for all $\lambda \in \mathbb{D}$.
Since the colligation Δ is unitary than $\Theta(\cdot) \in S_\kappa(\mathfrak{N}_2, \mathfrak{N}_1)$.

Definition 3.3. (see Langer (1971)) The operator function $\mathbb{R}_\lambda$, holomorphic in a neighborhood $\mathcal{O}$ of a point λ, is called a generalized resolvent of an isometric operator $V : \mathcal{H} \to \mathcal{H}$, if there exist a Pontryagin space $\widetilde{\mathcal{H}} \supset \mathcal{H}$ and a unitary extension $\widetilde{V} : \widetilde{\mathcal{H}} \to \widetilde{\mathcal{H}}$ of the operator V such that $\lambda \in \rho(\widetilde{V})$ and the equality

$$\mathbb{R}_\lambda = P_{\mathcal{H}} \left(\widetilde{V} - \lambda\right)^{-1} \upharpoonright \mathcal{H}, \lambda \in \rho(\widetilde{V}) \cap \mathcal{O}$$

holds; here $P_{\mathcal{H}}$ stands for the orthoprojector from $\widetilde{\mathcal{H}}$ onto $\mathcal{H}$.

4 BOUNDARY TRIPLETS IN A PONTYAGIN SPACE

In the case where $\mathcal{H}$ is a Hilbert space, the definition of the boundary triplet for an isometric operator was introduced in Malamud & Mogilevskii (2003).

4.1 Boundary triplets and extensions of an isometric operator in a Pontryagin space

Let $\mathcal{H}$ be a Pontryagin space with negative index κ, and let the operator $V : \mathcal{H} \to \mathcal{H}$ be an isometry in $\mathcal{H}$. By $\mathfrak{N}_1$ and $\mathfrak{N}_2$, we denote two auxiliary Hilbert spaces with inner products $(\cdot,\cdot)_{\mathfrak{N}_1}$ and $(\cdot,\cdot)_{\mathfrak{N}_2}$, respectively.

Definition 4.1. The collection $\Pi = \{\mathfrak{N}_1 \oplus \mathfrak{N}_2, \Gamma_1, \Gamma_2\}$ is called a boundary triplet of an isometric operator V, if

1) the following Green's generalized identity holds:

$$[f', g']_{\mathcal{H}} - [f, g]_{\mathcal{H}} = (\Gamma_1 \widehat{f}, \Gamma_1 \widehat{g})_{\mathfrak{N}_1} - (\Gamma_2 \widehat{f}, \Gamma_2 \widehat{g})_{\mathfrak{N}_2},$$

where $\widehat{f} = \begin{bmatrix} f \\ f' \end{bmatrix}, \widehat{g} = \begin{bmatrix} g \\ g' \end{bmatrix} \in V^{-[*]}$;

2) the mapping $\Gamma = (\Gamma_1, \Gamma_2)^T : V^{-[*]} \to \mathfrak{N}_1 \oplus \mathfrak{N}_2$ is surjective.

For an isometric operator, it is convenient to define the defect subspace $\mathfrak{N}_\lambda(V)$ as follows:

$$\mathfrak{N}_\lambda(V) := \ker\ \left(I - \lambda V^{[*]}\right) = \left\{ f_\lambda : \begin{bmatrix} f_\lambda \\ \lambda f_\lambda \end{bmatrix} \in V^{-[*]} \right\}, \quad \lambda \in \mathbb{C}.$$

We also set

$$\widehat{\mathfrak{N}}_\lambda(V) := \left\{ \begin{bmatrix} f_\lambda \\ \lambda f_\lambda \end{bmatrix} : f_\lambda \in \mathfrak{N}_\lambda(V) \right\}.$$

Let θ be a linear relation from $\mathfrak{N}_2$ to $\mathfrak{N}_1$. We define the extension V_θ of the operator V by the equality

$$V_\theta = \left\{ \widehat{f} \in V^{-[*]} : \begin{bmatrix} \Gamma_2 \widehat{f} \\ \Gamma_1 \widehat{f} \end{bmatrix} \in \theta \right\}.$$

The extension V_θ is, generally speaking, a linear relation in $\mathcal{H}$. Observe, that

$$V = \left\{ \widehat{f} \in V^{-[*]} : \Gamma_1 \widehat{f} = 0 \text{ and } \Gamma_2 \widehat{f} = 0 \right\}.$$

We define two extensions V_1 and V_2 of the operator V:

$$V_i = \left\{ \widehat{f} \in V^{-[*]} : \Gamma_i \widehat{f} = 0 \right\}, \quad i = 1, 2. \tag{4.1}$$

The extension V_1 is contractive in $\mathcal{H}$, whereas V_2 is an expansive relation in $\mathcal{H}$. As is known (Azizov & Iokhvidov (1989), p.186), the spectrum of the contractive extension V_1 contains at most κ points outside the unit disk $\mathbb{D}_e := \mathbb{C}\backslash\overline{\mathbb{D}}$, and the spectrum of the expanding extension V_2 contains at most κ points inside the unit disk $\mathbb{D}$.
Now define two sets of points:

$$\Lambda_1 = \{\lambda \in \mathbb{D}_e : \widehat{\mathfrak{N}}_\lambda(V) \cap V_1 \neq \{0\}\} = \sigma_p(V_1) \cap \mathbb{D}_e;$$

$$\Lambda_2 = \{\lambda \in \mathbb{D} : \widehat{\mathfrak{N}}_\lambda(V) \cap V_2 \neq \{0\}\} = \sigma_p(V_2) \cap \mathbb{D}.$$

Thus, each of the sets Λ_1 and Λ_2 contains at most κ points, and the sets

$$\mathcal{D}_1 := \mathbb{D}_e \setminus \Lambda_1 \text{ and } \mathcal{D}_2 := \mathbb{D} \setminus \Lambda_2 \tag{4.2}$$

are contained in the sets of regular points of these extensions.
The following Theorem taken from Publication III gives a connection between extensions of V and parameters θ.

Theorem 4.2. *Let the collection* $\Pi = \{\mathfrak{N}_1 \oplus \mathfrak{N}_2, \Gamma_1, \Gamma_2\}$ *be the boundary triplet for* V*, let* θ *be a linear relation from* $\mathfrak{N}_2$ *to* $\mathfrak{N}_1$*, and let* V_θ *be the corresponding extension of the operator* V*. Then*

(1) the inclusion $V_{\theta_1} \subset V_{\theta_2}$ *is equivalent to the inclusion* $\theta_1 \subset \theta_2$*;*

(2) $V_{\theta^{-*}} = V_\theta^{-[*]}$*;*

(3) V_θ *is a unitary extension of the operator* V*, iff* θ *is the graph of a unitary operator from* $\mathfrak{N}_2$ *to* $\mathfrak{N}_1$*;*

(4) V_θ *is an isometric extension of the operator* V*, iff* θ *is the graph f an isometric operator from* $\mathfrak{N}_2$ *to* $\mathfrak{N}_1$*;*

(5) V_θ *is a coisometric extension of the operator* V*, iff* θ *is the graph of a coisometric operator from* $\mathfrak{N}_2$ *to* $\mathfrak{N}_1$*;*

(6) V_θ *is a contraction, iff* θ *is a contaction;*

(7) V_θ *is an expansion, iff* θ *is an expansion.*

Note that, in assertions (3)–(6), the extension V_θ can be a linear relation with nontrivial multivalued part, whereas θ is the graph of an operator.

4.2 γ-fields and Weyl functions

The Weyl function of an isometric operator V allows one to describe the analytic properties of extensions of the operator V. We generalize the notion of the Weyl

function of an isometric operator V in a Hilbert space, which was introduced in Malamud & Mogilevskii (2003), for the case of the isometric operator V in a Pontryagin space $\mathcal{H}$ with negative index κ.
Let $\Pi = \{\mathfrak{N}_1 \oplus \mathfrak{N}_2, \Gamma_1, \Gamma_2\}$ be a boundary triplet for V, and let V_1 and V_2 be the extensions of the isometric operator V that were defined in (4.1). Then the mappings $\Gamma_j \upharpoonright \widehat{\mathfrak{N}}_\lambda(V) : \widehat{\mathfrak{N}}_\lambda(V) \to \mathfrak{N}_j \quad j = 1, 2$, are bounded and boundedly invertible for $\lambda \in \mathcal{D}_j$, see (4.2).
In this case, the operator-functions

$$\gamma_j(\lambda) := \pi_1 \widehat{\gamma}_j(\lambda) = \pi_1 \left(\Gamma_j \upharpoonright \widehat{\mathfrak{N}}_\lambda(V)\right)^{-1}$$

are well defined and called γ-fields for the l.r. $V^{-[*]}$.
By using γ-fields we can introduced two functions:

$$M_1(\lambda) := \Gamma_2 \widehat{\gamma}_1(\lambda), \quad \lambda \in \mathcal{D}_1;$$

$$M_2(\lambda) := \Gamma_1 \widehat{\gamma}_2(\lambda), \quad \lambda \in \mathcal{D}_2.$$

Observe, that the operator-function $M_2(\cdot)$ belongs to the class $S_\kappa(\mathfrak{N}_2, \mathfrak{N}_1)$.

Definition 4.3. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be as in (4.2). The operator-function defined by the equality

$$M(\lambda) = \begin{cases} M_1(\lambda), & \lambda \in \mathcal{D}_1 \\ M_2(\lambda), & \lambda \in \mathcal{D}_2 \end{cases},$$

is called the Weyl function of the operator $V : \mathcal{H} \to \mathcal{H}$ corresponding to the boundary triplet $\Pi = \{\mathfrak{N}_1 \oplus \mathfrak{N}_2, \Gamma_1, \Gamma_2\}$.

5 SUMMARIES OF THE ARTICLES

I. Completion, extension, factorization, and lifting of operators

In this article extensions of a result due to Yu. L. Shmul'yan on completions of nonnegative block operators are given. The extension of this fundamental result allows us generalized some well-known results of M. G. Kreĭn concerning the description of selfadjoint contractive extensions of a Hermitian contraction T_1 as well as the characterization of all nonnegative selfadjoint extensions $\widetilde{A}$ of a nonnegative operator A via the operator inequalities $A_K \leq \widetilde{A} \leq A_F$, where A_K and A_F are the Kreĭn-von Neumann extension and the Friedrichs extension of A.These generalizations concern the situation, where $\widetilde{A}$ is allowed to have a fixed number of negative eigenvalues. Furthermore, these new results are applied to solve some lifting problems for J-contractive operators in Hilbert, Pontryagin, and Kreĭn spaces. In addition, for instance a generalization of the well-known Douglas factorization of Hilbert space operators is derived. In the last part of this paper some very recent results concerning inequalities between semibounded selfadjoint relations and their inverses play a central role; such results are needed to treat the ordering of noncontractive selfadjoint operators under Cayley transforms properly.

II. Completion and extension of operators in Kreĭn spaces

This paper continuous the research carried out in Paper I. It develops further the approach based on completion problems by offering its natural extension to the Kreĭn and Pontryagin space setting. This allows us to generalize further the original results of M.G. Kreĭn about the description of selfadjoint contractive extension of a hermitian contraction. This generalization concerns the situation, where the selfadjoint operator A and extensions $\widetilde{A}$ belong to a Kreĭn space or a Pontryagin space and their defect operators are allowed to have a fixed number of negative eigenvalues. Also the result of Yu.L. Shmul'yan on completions of nonnegative block operators is extended for block operators with a fixed number of negative eigenvalues in a Kreĭn space.

III. On boundary triplets and generalized resolvents of an isometric operators in a Pontryagin space

In this paper the notions of boundary triplets and Weyl functions of an isometric operator V in the Pontryagin space setting are investigated. The results contain for instance a description of all proper extensions of the operator V and include a study of spectral properties of the unitary extensions of V. Formulas for canonical and generalized resolvents of the isometric operator V are established.

IV. Description of scattering matrices of unitary extensions of isometric operators in a Pontryagin space

An analog for the Kreĭn-Saakyan resolvent matrix theory is built in the setting of Pontryagin spaces. In particular, a new definition of a resolvent matrix of an isometric operator V is given and an abstract version of the Cristoffel-Darboux identity, known from the theory of orthogonal polynomials, is proven. By applying these results on resolvent matrices of an isometric operator V, a description of all scattering matrices of the operator V is established.

Bibliography

Adamjan, V., Arov, D., & Kreĭn, M.G. (1968). Infinite Hankel matrices and generalized problems of Caratheodory-Fejer and F. Riesz. *Funkcional. Anal. i Prilozen.* 2, 1, 1-19 (Russian).

Adamjan, V., Arov, D., & Kreĭn, M.G. (1971). Analytic properties of the Schmidt pairs of Hankel operator and the generalized Schur-Takagi problem. *Matem. Sb.* 86, 1, 31–73 (Russian).

Albeverio, S. & Kurasov, P. (1999). *Singular perturbations of differential operators: Solvable Schrödinger type operators.* London Math. Soc., Lecture Notes Series 271. Cambridge University Press.

Alpay D., Dijksma A., Rovnyak J., & de Snoo H.S.V.(1997). *Schur functions, operator colligations, and reproducing kernel Pontryagin spaces*. Oper. Theory: Adv. Appl., 96, Birkhäuser Verlag, Basel.

Ando, T. & Nishio, K. (1970). Positive self-adjoint extensions of positive symmetric operators. *Tohóku Math. J.* 22, 65–75.

Arov, D.Z. (1993). The generalized bitangent Caratheodory-Nevanlinna-Pick problem and (j, J_0)-inner matrix functions. *Izv. Ross. Akad. Nauk Ser. Mat.* 57, 1, 3–32.

Arov, D. Z. & Grossman, L. Z. (1992). Scattering matrices in the theory of unitary extension of isometric operators. *Math. Nachr.* 157, 105-123.

Azizov, T. Ya. and Iokhvidov, I. S. (1989). *Linear operators in spaces with indefinite metric*. John Wiley and Sons, New York.

Bognár, J. (1974). *Indefinite Inner Product Spaces*. Springer-Verlag, Berlin.

Brodskii, M. Sm. (1978). Unitary operator colligations and their characteristic functions. *Uspekhi Mat. Nauk* 33, 4, 141–168.

Calkin, J.W. (1939). Abstract symmetric boundary conditions. *TAMS* 45, 3, 369–442.

Coddington, E.A. and de Snoo, H.S.V. (1978). Positive selfadjoint extensions of positive symmetric subspaces. *Math. Z.* 159, 203–214.

Derkach V. (1995). On Weyl function and generalized resolvents of a Hermitian operator in a Kreĭn space. Integral Equations Operator Theory 23, 387–415.

Derkach V. (1999). On generalized resolvents of Hermitian relations in Kreĭn spaces. *J. Math. Sci.* 97, 5, 4420–4460.

Derkach V. A. (2001). On indefinite abstract interpolation problem. *Methods of Funct. Analysis and Topology* 7, 4, 87–100.

Derkach, V.A., Hassi, S., Malamud, M.M., & de Snoo, H.S.V. (2006). Boundary relations and their Weyl families. *Trans. Amer. Math. Soc.* 358, 5351-5400.

Derkach, V.A., Hassi, S., Malamud, M.M., & de Snoo, H.S.V. (2009). Boundary relations and generalized resolvents of symmetric operators. *Russ. J. Math. Phys.* 16, 1, 17-60.

Derkach, V.A., Hassi, S., Malamud, M.M., & de Snoo, H.S.V. (2012). Boundary triplets and Weyl functions. Recent developments. *London Math. Soc. Lecture Note Ser.* 404, 161-220.

Derkach, V.A. & Malamud, M.M. (1991). Generalized resolvents and the boundary value problems for hermitian operators with gaps. *J. Funct. Anal.* 95, 1–95.

Derkach, V.A. & Malamud, M.M. (1995). The extension theory of Hermitian operators and the moment problem *J. Math. Sci.* 73 (1995), 141–242.

Dijksma, A., Langer, H., & de Snoo, H.S.V. (1990). Generalized coresolvents of standard isometric operators and generalized resolvents of standard symmetric relations in Krein spaces. *Oper. Theory Adv. Appl.* 48, 261-274. Birkhäuser, Basel.

Gorbachuk, V.I. & Gorbachuk, M.L. (1990). *Boundary value problems for operator differential equations.* Kluwer Academic Publishers, Dordrecht, Boston, London.

Kostenko, A.S. & Malamud, M.M. (2010). 1–D Schrödinger operators with local point interactions on a discrete set. *J. Differential Equations* 249, 2, 253–304.

Kreĭn, M.G. (1944). On Hermitian operators with defect indices $(1, 1)$. *Dokl. Akad. Nauk SSSR* 43, 339–342.

Kreĭn, M.G. (1946) On resolvents of Hermitian operator with deficiency index (m, m). *Dokl. Akad. Nauk SSSR* 52, 657–660.

Kreĭn, M.G. (1947). Theory of selfadjoint extensions of semibounded operators and its applications, I. *Mat. Sb.*20, 3, 431–498.

Kreĭn, M. G. & Langer, G. K. (1971). The defect subspaces and generalized resolvents of a Hermitian operator in the space Π_κ. *Funkcional. Anal. i Prilozen* 5, 3, 54-69 (Russian).

Kreĭn, M. G. & Saakyan, Sh. N. (1966) Some new results in the theory of resolvents of Hermitian operators. *Dokl. Akad. Nauk SSSR* 169, 6, 1269–1272.

Langer, H. (1971). Generalized coresolvents of a π-isometric operator with unequal defect numbers. *Funkcional. Anal. i Prilozen.* 5, 4, 73-75.

Langer, H. & Sorjonen, P. Verallgemeinerte Resolventen Hermitescher und Isometrischer Operatoren im Pontrjaginraum. *Ann. Acad. Sci. Fenn. Ser. AI* 561, 3–45 (German).

Langer, H. and Textorius, B. (1977). On generalized resolvents and Q-functions of symmetric linear relations (subspaces) in Hilbert space. *Pacific J. Math.* 72, 1, 135–165.

Malamud, M.M. (1992). On a formula of the generalized resolvents of a nondensely defined Hermitian operator. *Ukr. Math. J.* 44, 1522–1547.

Malamud, M.M. & Mogilevskii V.I. (2003). Generalized resolvents of isometric operators. *Matem. Zametki* 43, 3, 460–466.

Malamud, M.M. & Mogilevskii V.I. (2005). Resolvent matrices and characteristic functions of extensions of isometric operators. *Dokl. Acad. Nauk* 405, 4, 454–461.

Nitz O. (2000). Generalized resolvents of isometric linear relations in Pontryagin spaces, I: Foundations. *Operator theory: Advances and Applications* 118, 303–319.

Nitz O. (2000). Generalized resolvents of isometric linear relations in Pontryagin spaces, II: Kreĭn–Langer formula *Methods of Functional Analysis and Topology* 6, 3, 72–96.

Reed, M. & Simon, B. (1975). *Methods of modern mathematical physics II: Fourier analysis, self-adjointness.* Academic Press, New York-San Francisco-London.

Pavlov, B.S. (1987). The theory of extensions and explicitly solvable models. *Uspekhi Mat. Nauk* 42, 99–131.

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Completion, extension, factorization, and lifting of operators

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Abstract The well-known results of M. G. Kreĭn concerning the description of selfadjoint contractive extensions of a hermitian contraction T_1 and the characterization of all nonnegative selfadjoint extensions $\widetilde{A}$ of a nonnegative operator A via the inequalities $A_K \leq \widetilde{A} \leq A_F$, where A_K and A_F are the Kreĭn–von Neumann extension and the Friedrichs extension of A, are generalized to the situation, where $\widetilde{A}$ is allowed to have a fixed number of negative eigenvalues. These generalizations are shown to be possible under a certain minimality condition on the negative index of the operators $I - T_1^* T_1$ and A, respectively; these conditions are automatically satisfied if T_1 is contractive or A is nonnegative, respectively. The approach developed in this paper starts by establishing first a generalization of an old result due to Yu. L. Shmul'yan on completions of nonnegative block operators. The extension of this fundamental result allows us to prove analogs of the above mentioned results of M. G. Kreĭn and, in addition, to solve some related lifting problems for J-contractive operators in Hilbert, Pontryagin and Kreĭn spaces in a simple manner. Also some new factorization results are derived, for instance, a generalization of the well-known Douglas factorization of Hilbert space operators. In the final steps of the treatment some very recent results concerning inequalities between semibounded selfadjoint relations and their inverses turn out to be central in order to treat the ordering of non-contractive selfadjoint operators under Cayley transforms properly.

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1 Introduction

Almost 70 years ago in his famous paper [47] M. G. Kreĭn proved that for a densely defined nonnegative operator A in a Hilbert space there are two extremal extensions of A, the Friedrichs (hard) extension A_F and the Kreĭn–von Neumann (soft) extension A_K, such that every nonnegative selfadjoint extension $\widetilde{A}$ of A can be characterized by the following two inequalities:

$$(A_F + a)^{-1} \leq (\widetilde{A} + a)^{-1} \leq (A_K + a)^{-1}, \quad a > 0.$$

To obtain such a description he used Cayley transforms of the form

$$T_1 = (I - A)(I + A)^{-1} T = (I - \widetilde{A})(I + \widetilde{A})^{-1},$$

to reduce the study of unbounded operators to the study of contractive selfadjoint extensions T of a hermitian nondensely defined contraction T_1. In the study of contractive selfadjoint extensions of T_1 he introduced a notion which is nowadays called "the shortening of a bounded nonnegative operator H to a closed subspace $\mathfrak{N}$" of $\mathfrak{H}$ as the (unique) maximal element in the set

$$\{ D \in [\mathfrak{H}] : 0 \leq D \leq H, \; \operatorname{ran} D \subset \mathfrak{N} \}, \tag{1}$$

which is denoted by $H_{\mathfrak{N}}$; cf. [3,4,57]. Here and in what follows the notation $[\mathfrak{H}_1, \mathfrak{H}_2]$ stands for the space of all bounded everywhere defined operators acting from $\mathfrak{H}_1$ to $\mathfrak{H}_2$; if $\mathfrak{H} = \mathfrak{H}_1 = \mathfrak{H}_2$ then the shorter notation $[\mathfrak{H}] = [\mathfrak{H}_1, \mathfrak{H}_2]$ is used. By means of shortening of operators he proved the existence of a minimal and maximal contractive extension T_m and T_M of T_1 and that T is a selfadjoint contractive extension of T_1 if and only if $T_m \leq T \leq T_M$.

Later the study of nonnegative selfadjoint extensions of $A \geq 0$ was generalized to the case of nondensely defined operators $A \geq 0$ by Ando and Nishio [5], as well as to the case of linear relations (multivalued linear operators) $A \geq 0$ by Coddington and de Snoo [22]. Further studies followed this work of M. G. Kreĭn; the approach in terms of "boundary conditions" to the extensions of a positive operator A was proposed by Vishik [63] and Birman [16]; an exposition of this theory based on the investigation of quadratic forms can be found from [2]. An approach to the extension theory of symmetric operators based on abstract boundary conditions was initiated even earlier by Calkin [21] under the name of reduction operators, and later, independently the technique of boundary triplets was introduced to formalize the study of boundary value problems in the framework of general operator theory; see [20,29,31,37,43,54]. Later the extension theory of unbounded symmetric Hilbert space operators and related resolvent formulas originating also from the work of Kreĭn [45,46], see also e.g. [52], was generalized to the spaces with indefinite inner products in the well-known series of papers by Langer and Kreĭn, see e.g. [49,50], and all of this has been further investigated, developed, and extensively applied in various other areas of mathematics and physics by numerous other researchers.

In spite of the long time span, natural extensions of the original results of Kreĭn in [47] seem not to be available in the literature. Obviously the most closely related result appears in Constantinescu and Gheondea [24], where for a given pair of a row operator $T_r = (T_{11}, T_{12}) \in [\mathfrak{H}_1 \oplus \mathfrak{H}'_1, \mathfrak{H}_2]$ and a column operator $T_c = \mathrm{col}\,(T_{11}, T_{21}) \in [\mathfrak{H}_1, \mathfrak{H}_2 \oplus \mathfrak{H}'_2]$ the problem for determining all possible operators $\widetilde{T} \in [\mathfrak{H}_1 \oplus \mathfrak{H}'_1, \mathfrak{H}_2 \oplus \mathfrak{H}'_2]$ acting from the Hilbert space $\mathfrak{H}_1 \oplus \mathfrak{H}'_1$ to the Hilbert space $\mathfrak{H}_2 \oplus \mathfrak{H}'_2$ such that

$$P_{\mathfrak{H}_2}\widetilde{T} = T_r, \quad \widetilde{T}{\restriction}\,\mathfrak{H}_1 = T_c,$$

and such that the following negative index (number of negative eigenvalues) conditions are satisfied

$$\kappa_1 := \nu_-(I - \widetilde{T}^*\widetilde{T}) = \nu_-(I - T_c^*T_c), \quad \kappa_2 := \nu_-(I - \widetilde{T}\widetilde{T}^*) = \nu_-(I - T_rT_r^*),$$

is considered. The problem was solved in [24, Theorem 5.1] under the condition $\kappa_1, \kappa_2 < \infty$. In the literature cited therein appears also a reference to an unpublished manuscript [53] by H. Langer and B. Textorius, where a similar problem for a given bounded hermitian column operator T has been investigated; see [53, Theorems 1.1, 2.8][1] and [24, Section 6]. However, in these papers the existence of possible extremal extensions in the solution set in the spirit of [47], when it is nonempty, have not been investigated. Also possible investigations of analogous results for unbounded symmetric operators with a fixed negative index seem to be unavailable in the literature.

In this paper we study classes of "quasi-contractive" symmetric operators T_1 with $\nu_-(I - T_1^*T_1) < \infty$ as well as "quasi-nonnegative" operators A with $\nu_-(A) < \infty$ and the existence and description of all possible selfadjoint extensions T and $\widetilde{A}$ of them which preserve the given negative indices $\nu_-(I - T^2) = \nu_-(I - T_1^*T_1)$ and $\nu_-(\widetilde{A}) = \nu_-(A)$, and prove precise analogs of the above mentioned results of M. G. Kreĭn under a minimality condition on the negative indices $\nu_-(I - T_1^*T_1)$ and $\nu_-(A)$, respectively. It is an unexpected fact that when there is a solution then the solution set still contains a minimal solution and a maximal solution which then describe the whole solution set via two operator inequalities, just as in the original paper of M. G. Kreĭn. The approach developed in this paper differs from the approach in [47]. In fact, technique based on nonnegative completions of operators appearing in papers by Kolmanovich and Malamud [44] and Hassi et al. [39] will be successfully generalized. In particular, we introduce a new class of completion problems for Hilbert space operators, whose solutions evidently admit a wider scope of applications than what is appearing in the present paper.

The starting point in our approach is to establish a generalization of an old result due to Shmul'yan [59] on completions of nonnegative block operators where the result

[1] After the Math ArXiv version of the present paper we inquired contents of that work from H. Langer who then kindly provided us their initial work in [53].

was applied for introducing so-called Hellinger operator integrals. Our extension of this fundamental result is given in Sect. 2; see Theorem 1 (for the case $\kappa < \infty$) and Theorem 2 (for the case $\kappa = \infty$). Obviously these two results, already in view of the various consequences appearing in later sections, may be considered as being most useful inventions in the present paper with further possible applications in problems appearing also elsewhere (see e.g. [4,6,27,28,58]).

In this paper we will extensively apply Theorem 1. In Sect. 3 this result is specialized to a class of block operators to characterize occurrence of a minimal negative index for the so-called Schur complement, see Theorem 3. This result can be also viewed as a factorization result and, in fact, it yields a generalization of the well-known Douglas factorization of Hilbert space operators in [32], see Proposition 1, which is completed by a generalization of Sylvester's criterion on additivity of inertia on Schur complements in Proposition 2. In Sect. 4, Theorem 1, or its special case Theorem 3, is applied to solve lifting problems for J-contractive operators in Hilbert, Pontryagin and Kreĭn spaces in a new simple way, the most general version of which is formulated in Theorem 4: this result was originally proved in Constantinescu and Gheondea [23, Theorem 2.3] with the aid of [13, Theorem 5.3]; for special cases, see also Dritschel and Rovnyak [33,34]. In the Hilbert space case the problem has been solved in [12,25,62], further proofs and facts can be found e.g. from [8,10,19,44,55].

Section 5 contains the extension of the fundamental result of Kreĭn in [47], see Theorem 5, which characterizes the existence and gives a description of all selfadjoint extensions T of a bounded symmetric operator T_1 satisfying the following minimal index condition $\nu_-(I - T^2) = \nu_-(I - T_{11}^2)$ by means of two extreme extensions via $T_m \le T \le T_M$. In Sect. 6 selfadjoint extensions of unbounded symmetric operators, and symmetric relations, are studied under a similar minimality condition on the negative index $\nu_-(A)$; the main result there is Theorem 8. It is a natural extension of the corresponding result of Kreĭn in [47]. The treatment here uses Cayley type transforms and hence is analogous to that in [47]. However, the existence of two extremal extensions in this setting and the validity of all the operator inequalities appearing therein depend essentially on so-called "antitonicity results" proved only very recently for semibounded selfadjoint relations in [15] concerning correctness of the implication $H_1 \le H_2 \Rightarrow H_1^{-1} \ge H_2^{-1}$ in the case that H_1 and H_2 have some finite negative spectra. In this section analogs of the so-called Kreĭn's uniqueness criterion for the equality $T_m = T_M$ are also established.

2 A completion problem for block operators

By definition the modulus $|C|$ of a closed operator C is the nonnegative selfadjoint operator $|C| = (C^*C)^{1/2}$. Every closed operator admits a polar decomposition $C = U|C|$, where U is a (unique) partial isometry with the initial space $\overline{\operatorname{ran}}\,|C|$ and the final space $\overline{\operatorname{ran}}\,C$, cf. [42]. For a selfadjoint operator $H = \int_{\mathbb{R}} t\, dE_t$ in a Hilbert space $\mathfrak{H}$ the partial isometry U can be identified with the signature operator, which can be taken to be unitary: $J = \operatorname{sign}(H) = \int_{\mathbb{R}} \operatorname{sign}(t)\, dE_t$, in which case one should define $\operatorname{sign}(t) = 1$ if $t \ge 0$ and otherwise $\operatorname{sign}(t) = -1$.

2.1 Completion to operator blocks with finite negative index

The following theorem solves a completion problem for a bounded incomplete block operator A^0 of the form

$$A^0 = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & * \end{pmatrix} \begin{pmatrix} \mathfrak{H}_1 \\ \mathfrak{H}_2 \end{pmatrix} \to \begin{pmatrix} \mathfrak{H}_1 \\ \mathfrak{H}_2 \end{pmatrix} \tag{2}$$

in the Hilbert space $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$.

Theorem 1 *Let $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ be an orthogonal decomposition of the Hilbert space $\mathfrak{H}$ and let A^0 be an incomplete block operator of the form (2). Assume that $A_{11} = A_{11}^*$ and $A_{21} = A_{12}^*$ are bounded, $\nu_-(A_{11}) = \kappa < \infty$, where $\kappa \in \mathbb{Z}_+$, and let $J = \operatorname{sign}(A_{11})$ be the (unitary) signature operator of A_{11}. Then:*

(1) *There exists a completion $A \in [\mathfrak{H}]$ of A^0 with some operator $A_{22} = A_{22}^* \in [\mathfrak{H}_2]$ such that $\nu_-(A) = \nu_-(A_{11}) = \kappa$ if and only if*

$$\operatorname{ran} A_{12} \subset \operatorname{ran} |A_{11}|^{1/2}. \tag{3}$$

(2) *If (3) is satisfied, then the operator $S = |A_{11}|^{[-1/2]} A_{12}$, where $|A_{11}|^{[-1/2]}$ denotes the (generalized) Moore–Penrose inverse of $|A_{11}|^{1/2}$, is well defined and $S \in [\mathfrak{H}_2, \mathfrak{H}_1]$. Moreover, $S^* J S$ is the smallest operator in the solution set*

$$\mathcal{A} := \{A_{22} = A_{22}^* \in [\mathfrak{H}_2] : A = (A_{ij})_{i,j=1}^2 : \nu_-(A) = \kappa\} \tag{4}$$

and this solution set admits a description as the (semibounded) operator interval given by

$$\mathcal{A} = \{A_{22} \in [\mathfrak{H}_2] : A_{22} = S^* J S + Y,\ Y = Y^* \geq 0\}.$$

Proof (i) Assume that there exists a completion $A_{22} \in \mathcal{A}$. Let $\lambda_\kappa \leq \lambda_{\kappa-1} \leq \cdots \leq \lambda_1 < 0$ be all the negative eigenvalues of A_{11} and let ε be such that $|\lambda_1| > \varepsilon > 0$. Then $0 \in \rho(A_{11} + \varepsilon)$ and hence one can write

$$\begin{aligned} &\begin{pmatrix} I & 0 \\ -A_{21}(A_{11}+\varepsilon)^{-1} & I \end{pmatrix} \begin{pmatrix} A_{11}+\varepsilon & A_{12} \\ A_{21} & A_{22}+\varepsilon \end{pmatrix} \begin{pmatrix} I & -(A_{11}+\varepsilon)^{-1}A_{12} \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} A_{11}+\varepsilon & 0 \\ 0 & A_{22}+\varepsilon - A_{21}(A_{11}+\varepsilon)^{-1}A_{12} \end{pmatrix} \end{aligned} \tag{5}$$

The operator in the righthand side of (5) has κ negative eigenvalues if and only if

$$A_{21}(A_{11}+\varepsilon)^{-1}A_{12} \leq A_{22} + \varepsilon \tag{6}$$

or equivalently

$$\int_{-\|A_{11}\|}^{\|A_{11}\|} (t+\varepsilon)^{-1} d\|E_t A_{12} f\|^2 \le \varepsilon \|f\|^2 + (A_{22} f, f), \tag{7}$$

where E_t is the spectral family of A_{11} and $f \in \mathfrak{H}_2$. We rewrite (7) in the form

$$\begin{aligned}&\int_{[-\|A_{11}\|,0)} (t+\varepsilon)^{-1} d\|E_t A_{12} f\|^2 + \int_{[0,\|A_{11}\|]} (t+\varepsilon)^{-1} d\|E_t A_{12} f\|^2 \\ &\le \varepsilon \|f\|^2 + (A_{22} f, f),\end{aligned}$$

This yields the estimate

$$\int_{[0,\|A_{11}\|]} (t+\varepsilon)^{-1} d\|E_t A_{12} f\|^2 \le \varepsilon \|f\|^2 + (A_{22} f, f) - \frac{1}{\lambda_1+\varepsilon} \|A_{12} f\|^2. \tag{8}$$

By letting $\varepsilon \searrow 0$ in (8) the monotone convergence theorem implies that

$$P_+ A_{12} f \in \operatorname{ran} A_{11+}^{1/2} \subset \operatorname{ran} |A_{11}|^{1/2}$$

for all $f \in \mathfrak{H}_2$; here $A_{11+} = \int_{[0,\|A_{11}\|]} t\, dE_t$ stands for the nonnegative part of A_{11} and P_+ is the orthogonal projection onto the corresponding closed subspace $\overline{\operatorname{ran}}\, A_{11+} = \int_{[0,\|A_{11}\|]} dE_t$. Since $\operatorname{ran}(I-P_+)$ is the κ-dimensional spectral subspace of A_{11} corresponding to its negative spectrum, one concludes that

$$(I-P_+) A_{12} f \in \operatorname{ran} A_{11} \subset \operatorname{ran} |A_{11}|^{1/2}$$

for all $f \in \mathfrak{H}_2$. Therefore, $\operatorname{ran} A_{12} \subset \operatorname{ran} |A_{11}|^{1/2}$.

Conversely, if $\operatorname{ran} A_{12} \subset \operatorname{ran} |A_{11}|^{1/2}$, then the operator $S := |A_{11}|^{[-1/2]} A_{12}$ is well defined, closed and bounded, i.e., $S \in [\mathfrak{H}_2, \mathfrak{H}_1]$. Since $A_{12} = |A_{11}|^{1/2} S$, it follows from $A_{21} = S^* |A_{11}|^{1/2}$ and

$$\begin{pmatrix} |A_{11}|^{1/2} \\ S^* J \end{pmatrix} J \left(|A_{11}|^{1/2} \;\; JS \right) : \nu_-(A) = \kappa, \tag{9}$$

that the operator $A_{22} = S^* J S$ gives a completion for A^0.

(ii) The proof of (i) shows that $A_{21} = S^* |A_{11}|^{1/2}$ is well defined and that $S^* J S \in [\mathfrak{H}_2]$ gives a solution to the completion problem (2). Now

$$s - \lim_{\varepsilon \searrow 0} A_{21} (A_{11} + \varepsilon)^{-1} A_{12} = s - \lim_{\varepsilon \searrow 0} S^* |A_{11}|^{1/2} (A_{11} + \varepsilon)^{-1} |A_{11}|^{1/2} S = S^* J S$$

and if A_{22} is an arbitrary operator in the set (4), then by letting $\varepsilon \searrow 0$ one concludes that $S^* J S \le A_{22}$. Therefore, $S^* J S$ satisfies the desired minimality property.

To prove the last statement assume that $Y \in [\mathfrak{H}_2]$ and that $Y \geq 0$. Then $A_{22} = S^* J S + Y$ inserted in A^0 defines a block operator $A_Y \geq A_{\min}$. In particular, $\nu_-(A_Y) \leq \nu_-(A_{\min}) = \kappa < \infty$. On the other hand, it is clear from the formula

$$A_Y = \begin{pmatrix} |A_{11}|^{1/2} \\ S^* J \end{pmatrix} J \left(|A_{11}|^{1/2} \; J S \right) + \begin{pmatrix} 0 & 0 \\ 0 & Y \end{pmatrix} \tag{10}$$

that the κ-dimensional eigenspace corresponding to the negative eigenvalues of A_{11} is A_Y-negative and, hence, $\nu_-(A_Y) \geq \kappa$. Therefore, $\nu_-(A_Y) = \kappa$ and $Y \in \mathcal{A}$.

Notice that in the factorization $A_{12} = |A_{11}|^{1/2} S$, S is uniquely determined under the condition $\operatorname{ran} S \subset \overline{\operatorname{ran}}\, A_{11}$ (which implies that $\ker A_{12} = \ker S$); cf. [32].

In the case that $\kappa = 0$, the result in Theorem 1 reduces to the well-known criterion concerning completion of an incomplete block operator to a nonnegative operator; cf. [59]. In the case of matrices acting on a finite dimensional Hilbert space, the result with $\kappa > 0$ has been proved very recently in the appendix of [28], where it was applied in solving indefinite truncated moment problems. In the present paper Theorem 1 will be one of the main tools for further investigations.

2.2 Completion to operator blocks with an infinite negative index.

The completion result in Theorem 1 is of some general interest already by the substantial number of its applications known in the case of nonnegative operators. In this section the completion problem is treated in the case that $\kappa = \infty$. For this purpose some further notions will be introduced.

Recall that a subspace $\mathfrak{M} \subset \mathfrak{H}$ is said to be uniformly A-negative, if there exists a positive constant $\nu > 0$ such that $(Af, f) \leq -\nu \|f\|^2$ for all $f \in \mathfrak{M}$. It is maximal uniformly A-negative, if $\mathfrak{M}$ has no proper uniformly A-negative extension. The completion problem is now extended by claiming from the completions the following maximality property:

$$\text{There exists a subspace } \mathfrak{M} \subset \mathfrak{H}_1 \text{ which is maximal uniformly A-negative.} \tag{11}$$

Theorem 2 *Let A^0 be an incomplete block operator of the form (2) in the Hilbert space $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$. Let $A_{11} = A_{11}^*$ and $A_{21} = A_{12}^*$ be bounded, let $J = \operatorname{sign}(A_{11})$ be the (unitary) signature operator of A_{11}, and, in addition, assume that there is a spectral gap $(-\delta, 0) \subset \rho(A_{11})$, $\delta > 0$. Then:*

(i) *There exists a completion $A \in [\mathfrak{H}]$ of A^0 with some operator $A_{22} = A_{22}^*$ satisfying the condition (11) if and only if*

$$\operatorname{ran} A_{12} \subset \operatorname{ran} |A_{11}|^{1/2}.$$

(ii) *If the condition in (i) is satisfied, then $S = |A_{11}|^{[-1/2]} A_{12}$, where $|A_{11}|^{[-1/2]}$ denotes the (generalized) Moore–Penrose inverse of $|A_{11}|^{1/2}$, is well defined and*

$S \in [\mathfrak{H}_2, \mathfrak{H}_1]$. *Moreover,* S^*JS *is the smallest operator in the solution set*

$$\mathcal{A} := \{A_{22} = A_{22}^* \in [\mathfrak{H}_2] : A = (A_{ij})_{i,j=1}^2 \text{ satisfies } (11)\}$$

and this solution set admits a description as the (*semibounded*) *operator interval given by*

$$\mathcal{A} = \{A_{22} \in [\mathfrak{H}_2] : A_{22} = S^*JS + Y,\ Y = Y^* \geq 0\}.$$

Proof To prove this result suitable modifications in the proof of Theorem 1 are needed. (i) First assume that $A_{22} \in \mathcal{A}$ gives a desired completion for A^0. If $\varepsilon \in (0, \delta)$ then $0 \in \rho(A_{11} + \varepsilon)$ and therefore the block operator (A_{ij}) satisfies the formula (5). We claim that the condition (11) implies the inequality (6) for all sufficiently small values $\varepsilon > 0$. To see this let $\mathfrak{M} \subset \mathfrak{H}_1$ be a subspace for which the condition (11) is satisfied. Then $(A_{11}f, f) \leq -\nu\|f\|^2$ for some fixed $\nu > 0$ and for all $f \in \mathfrak{M}$. Assume that for some $0 < \varepsilon_0 < \min\{\nu, \delta\}$ (6) is not satisfied. Then $((A_{22} + \varepsilon_0 - A_{21}(A_{11} + \varepsilon_0)^{-1}A_{12})v_0, v_0) < 0$ holds for some vector $v_0 \in \mathfrak{H}_2$. Define $\mathfrak{L} = W_{\varepsilon_0}^{-1}(\mathfrak{M} + \text{span}\,\{v_0\})$, where

$$W_{\varepsilon_0} = \begin{pmatrix} I & -(A_{11} + \varepsilon_0)^{-1}A_{12} \\ 0 & I \end{pmatrix}.$$

Clearly, W_{ε_0} is bounded with bounded inverse and it maps $\mathfrak{M}$ bijectively onto $\mathfrak{M}$, so that $\mathfrak{L}$ is a 1-dimensional extension of $\mathfrak{M}$. It follows from (5) that for all $f \in \mathfrak{L}$,

$$(Af, f) + \varepsilon_0\|f\|^2 = \left(\begin{pmatrix} A_{11} + \varepsilon_0 & 0 \\ 0 & A_{22} + \varepsilon_0 - A_{21}(A_{11} + \varepsilon_0)^{-1}A_{12} \end{pmatrix} u, u\right) < 0,$$

where $u = W_{\varepsilon_0}f \in \mathfrak{M} + \text{span}\,\{v_0\}$. Therefore, $\mathfrak{L}$ is a proper uniformly A-negative extension of $\mathfrak{M}$; a contradiction, which shows that (6) holds for all $0 < \varepsilon < \min\{\nu, \delta\}$. Then, as in the proof of Theorem 1 it is seen that $\text{ran}\, A_{12} \subset \text{ran}\, |A_{11}|^{1/2}$; note that in the estimate (8) λ_1 is to be replaced by $-\delta$.

Conversely, if $\text{ran}\, A_{12} \subset \text{ran}\, |A_{11}|^{1/2}$, then $S = |A_{11}|^{[-1/2]}A_{12} \in [\mathfrak{H}_2, \mathfrak{H}_1]$ and the block operator A in (9) gives a completion. To prove that A satisfies (11) observe that if $\mathfrak{M}$ is a uniformly A-negative subspace in $\mathfrak{H}$, then $\left(|A_{11}|^{1/2}\ JS\right)$ maps it bijectively onto a uniformly J-negative subspace in $\mathfrak{H}_1$. The spectral subspace corresponding to the negative spectrum of A_{11} is maximal uniformly J-negative in $\mathfrak{H}_1$ and also uniformly A-negative in $\mathfrak{H}$. By the above mapping property this subspace must be maximal uniformly A-negative in $\mathfrak{H}$.

(ii) If $A_{22} = A_{22}^*$ defines a completion $A \in [\mathfrak{H}]$ of A^0 such that (11) is satisfied then by the proof of (i) the inequality (6) holds for all sufficiently small values $\varepsilon > 0$. Now the minimality property of S^*JS can be obtained in the same manner as in Theorem 1.

As to the last statement again for every $Y \in [\mathfrak{H}_2]$, $Y \geq 0$, the block operator A_Y defined in the proof of Theorem 1 satisfies $A_Y \geq A_{\min}$. Hence, every uniformly A_Y-negative subspace is also uniformly $A_{\min}$-negative. Now it follows from the formula (10) that the spectral subspace corresponding to the negative spectrum of A_{11}, which is maximal uniformly $A_{\min}$-negative, is also maximal uniformly A_Y-negative. Hence, A_Y satisfies (11) and $Y \in \mathcal{A}$.

3 Some factorizations of operators with finite negative index

Theorems 1 and 2 contain a valuable tool in solving a couple of other problems, which initially do not occur as a completion problem of some symmetric incomplete block operator. In this section it is shown that Theorem 1 (a) can be used to characterize the existence of certain J-contractive factorizations of operators via a minimal index condition; (b) implies an extension of the well-known Douglas factorization result with a certain specification to the Bognár–Krámli factorization; (c) yields an extension of a factorization result of Shmul'yan for J-bicontractions; (d) allows an extension of a classical Sylvester's law of inertia of a block operator, which is originally used in characterizing nonnegativity of a bounded block operator via Schur complement.

Some simple inertia formulas are now recalled. The factorization $H = B^*EB$ clearly implies that $\nu_\pm(H) \le \nu_\pm(E)$. If H_1 and H_2 are selfadjoint operators, then

$$H_1 + H_2 = \begin{pmatrix} I \\ I \end{pmatrix}^* \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} \begin{pmatrix} I \\ I \end{pmatrix}$$

shows that $\nu_\pm(H_1 + H_2) \le \nu_\pm(H_1) + \nu_\pm(H_2)$. Consider the selfadjoint block operator $H \in [\mathfrak{H}_1 \oplus \mathfrak{H}_2]$ of the form

$$H = \begin{pmatrix} A & B^* \\ B & J_2 \end{pmatrix}, \tag{12}$$

where $J_2 = J_2^* = J_2^{-1}$. By applying the above mentioned inequalities shows that

$$\nu_\pm(A) \le \nu_\pm(A - B^*J_2B) + \nu_\pm(J_2). \tag{13}$$

Assuming that $\nu_-(A - B^*J_2B)$ and $\nu_-(J_2)$ are finite, the question when $\nu_-(A)$ attains its maximum in (13), or equivalently, $\nu_-(A - B^*J_2B) \ge \nu_-(A) - \nu_-(J_2)$ attains its minimum, turns out to be of particular interest. The next result characterizes this situation as an application of Theorem 1. Recall that if $A = J_A|A|$ is the polar decomposition of A, then one can interpret $\mathfrak{H}_A = (\overline{\operatorname{ran}}\, A, J_A)$ as a Kreĭn space generated on $\overline{\operatorname{ran}}\, A$ by the fundamental symmetry $J_A = \operatorname{sgn}(A)$.

Theorem 3 *Let $A \in [\mathfrak{H}_1]$ be selfadjoint, $B \in [\mathfrak{H}_1, \mathfrak{H}_2]$, $J_2 = J_2^* = J_2^{-1} \in [\mathfrak{H}_2]$, and assume that $\nu_-(A), \nu_-(J_2) < \infty$. If the equality*

$$\nu_-(A) = \nu_-(A - B^*J_2B) + \nu_-(J_2) \tag{14}$$

holds, then $\operatorname{ran} B^* \subset \operatorname{ran} |A|^{1/2}$ and $B^* = |A|^{1/2}K$ for a unique operator $K \in [\mathfrak{H}_2, \mathfrak{H}_A]$ which is J-contractive: $J_2 - K^*J_AK \ge 0$.

Conversely, if the equality $B^* = |A|^{1/2}K$ holds for some J-contractive operator $K \in [\mathfrak{H}_2, \overline{\operatorname{ran}}\, A]$, then the equality (14) is satisfied.

Proof Assume that (14) is satisfied. The factorization

$$H = \begin{pmatrix} A & B^* \\ B & J_2 \end{pmatrix} = \begin{pmatrix} I & B^*J_2 \\ 0 & I \end{pmatrix} \begin{pmatrix} A - B^*J_2B & 0 \\ 0 & J_2 \end{pmatrix} \begin{pmatrix} I & 0 \\ J_2B & I \end{pmatrix}$$

shows that $\nu_-(H) = \nu_-(A - B^*J_2B) + \nu_-(J_2)$, which combined with the equality (14) gives $\nu_-(H) = \nu_-(A)$. Therefore, by Theorem 1 one has $\operatorname{ran} B^* \subset \operatorname{ran} |A|^{1/2}$ and this is equivalent to the existence of a unique operator $K \in [\mathfrak{H}_2, \overline{\operatorname{dom} A}]$ such that $B^* = |A|^{1/2}K$; i.e. $K = |A|^{[-1/2]}B^*$. Furthermore, $K^*J_AK \leq J_2$ by the minimality property of K^*J_AK in Theorem 1, in other words K is a J-contraction.

Converse, if $B^* = |A|^{1/2}K$ for some J-contraction $K \in [\mathfrak{H}_2, \overline{\operatorname{dom} A}]$, then clearly $\operatorname{ran} B^* \subset \operatorname{ran} |A|^{1/2}$. By Theorem 1 the completion problem for H^0 has solutions with the minimal solution S^*J_AS, where

$$S = |A|^{[-1/2]}B^* = |A|^{[-1/2]}|A|^{1/2}K = K.$$

Furthermore, by J-contractivity of K one has $K^*J_AK \leq J_2$, i.e. J_2 is also a solution and thus $\nu_-(H) = \nu_-(A)$ or, equivalently, the equality (14) is satisfied.

While Theorem 3 is obtained as a direct consequence of Theorem 1 it will be shown in the next section that this result yields simple solutions to a wide class of lifting problems for contractions in Hilbert, Pontryagin and Kreĭn space settings.

Before deriving the next result some inertia formulas for a class of selfadjoint block operators are recalled. Consider the following two representations

$$\begin{pmatrix} J_1 & T^* \\ T & J_2 \end{pmatrix} = \begin{pmatrix} I & 0 \\ TJ_1 & I \end{pmatrix} \begin{pmatrix} J_1 & 0 \\ 0 & J_2 - TJ_1T^* \end{pmatrix} \begin{pmatrix} I & J_1T^* \\ 0 & I \end{pmatrix}$$
$$= \begin{pmatrix} I & T^*J_2 \\ 0 & I \end{pmatrix} \begin{pmatrix} J_1 - T^*J_2T & 0 \\ 0 & J_2 \end{pmatrix} \begin{pmatrix} I & 0 \\ J_2T & I \end{pmatrix},$$

where $J_i = J_i^* = J_i^{-1}$, $i = 1, 2$. Since here the triangular operators are bounded with bounded inverse, one concludes that $\operatorname{ran}(J_2 - TJ_1T^*)$ is closed if and only if $\operatorname{ran}(J_1 - T^*J_2T)$ is closed. Furthermore, one gets the following inertia formulas; cf. e.g. [13, Proposition 3.1].

Lemma 1 *With the above notations one has*

$$\nu_\pm(J_1 - T^*J_2T) + \nu_\pm(J_2) = \nu_\pm(J_2 - TJ_1T^*) + \nu_\pm(J_1),$$

$$\nu_0(J_1 - T^*J_2T) = \nu_0(J_2 - TJ_1T^*).$$

The next result contains two general factorization results: assertion (i) contains an extension of the well-known Douglas factorization, see [32,35], and assertion (ii) is a specification of the so-called Bognár–Krámli factorization, see [18]: $A = B^*J_2B$ holds for some bounded operator B if and only if $\nu_\pm(J_2) \geq \nu_\pm(A)$.

Proposition 1 *Let A, B, and J_2 be as in Theorem 3, and let $\nu_-(A) = \nu_-(J_2) < \infty$. Then*:

(i) *The inequality*

$$A \geq B^* J_2 B \tag{15}$$

holds if and only if $B = C|A|^{1/2}$ for some J-contractive operator $C \in [\mathfrak{H}_A, \mathfrak{H}_2]$; in this case C is unique and, in addition, J-bicontractive, i.e., $J_A - C^* J_2 C \geq 0$ and $J_2 - C J_A C^* \geq 0$.

(ii) *The equality*

$$A = B^* J_2 B \tag{16}$$

holds if and only if $B = C|A|^{1/2}$ for some J-isometric operator $C \in [\mathfrak{H}_A, \mathfrak{H}_2]$; again C is unique. In addition, C is unitary if and only if ran B *is dense in $\mathfrak{H}_2$.*

Proof (i) The inequality (15) means that $\nu_-(A - B^* J_2 B) = 0$. Hence the assumption $\nu_-(A) = \nu_-(J_2) < \infty$ implies the equality (14). Therefore, the desired factorization for B is obtained from Theorem 3. Conversely, if $B = C|A|^{1/2}$ for some J-contractive operator C then (14) holds by Theorem 3 and the assumption $\nu_-(A) = \nu_-(J_2) < \infty$ implies that $\nu_-(A - B^* J_2 B) = 0$.

The fact that C is actually J-bicontractive follows directly from Lemma 1.

(ii) Assume that (16) holds. Then by part (i) it remains to prove that in the factorization $B = C|A|^{1/2}$ the operator C is isometric. Substituting $B = C|A|^{1/2}$ into (16) gives

$$A = |A|^{1/2} C^* J_2 C |A|^{1/2}.$$

Since dom C, ran $C^* \subset \overline{\operatorname{ran}}\, A$ and $A = |A|^{1/2} J_A |A|^{1/2}$, the previous identity implies the equality $J_A = C^* J_2 C$, i.e., C is J-isometric. Conversely, if C is J-isometric then clearly (16) holds.

Since $B = C|A|^{1/2}$ and $C \in [\mathfrak{H}_A, \mathfrak{H}_2]$, it is clear that B has dense range in $\mathfrak{H}_2$ precisely when the range of C is dense in $\mathfrak{H}_2$. The (Kreĭn space) adjoint is a bounded operator with dom $C^{[*]} = \mathfrak{H}_2$. By isometry one has $C^{-1} \subset C^{[*]}$, and thus C^{-1} is also bounded, densely defined and closed. Thus, the equality $C^{-1} = C^{[*]}$ prevails, i.e., C is J-unitary. Conversely, if C is unitary then $C^{-1} = C^{[*]}$ holds and ran C = dom $C^{[*]} = \mathfrak{H}_2$. Consequently, ran B = ran $C|A|^{1/2}$ is dense in $\mathfrak{H}_2$.

If, in particular, $\nu_-(A) = \nu_-(J_2) = 0$ then $0 \leq A \leq B^* B$ and Proposition 1 combined with Theorem 1 yields the factorization and range inclusion results proved in [32, Theorem 1] with A replaced by $A^* A$. In particular, notice that if ran $B^* \subset$ ran $|A|^{1/2}$, then already Theorem 1 alone implies that $S = |A|^{[-1/2]} B^*$ is bounded and hence $B^* B = |A|^{1/2} S S^* |A|^{1/2} \leq \|S\|^2 A$.

Assertions in part (ii) of Corollary 1 can be found in the literature with a different proof. In fact, the first statement in (ii) appears in [13, Proposition 2.1,Corollary 2.6] while the second statement in (ii) is proved in [23, Corollary 1.3]. Another extension for Douglas' factorization result can be found from [58].

For a general treatment of isometric (not necessarily densely defined) operators and isometric relations appearing in the proof of Proposition 1 the reader is referred to [14], [26, Section 2], and [27].

A slightly different viewpoint to Proposition 1 gives the following statement, which can be viewed as an extension of a theorem by Shmul'yan, see [60, Theorem 3], on the factorization of bicontractions on Kreĭn spaces; for a related abstract Leech theorem, see [34, Section 3.4].

Corollary 1 *Let* $A \in [\mathfrak{H}_1]$ *be selfadjoint, let* $B \in [\mathfrak{H}_1, \mathfrak{H}_2]$, *and let* $J_2 = J_2^* = J_2^{-1} \in [\mathfrak{H}_2]$ *with* $\nu_-(J_2) < \infty$. *Then*:

(i)

$$A \geq B^* J_2 B \quad \text{and} \quad \nu_-(A) = \nu_-(J_2)$$

if and only if $B = C|A|^{1/2}$ *for some* J*-bicontractive operator* $C \in [\mathfrak{H}_A, \mathfrak{H}_2]$; *in this case* C *is unique.*

(ii)

$$A = B^* J_2 B \quad \text{and} \quad \nu_-(A) = \nu_-(J_2)$$

if and only if $B = C|A|^{1/2}$ *for some* J*-bicontractive operator* C *which is also* J*-isometric, i.e.,* $J_A - C^* J_2 C = 0$ *and* $J_2 - C J_A C^* \geq 0$; *again* C *is unique.*

Proof Observe that if C is J-bicontractive, then an application of Lemma 1 shows that $\nu_-(J_2) = \nu_-(J_A) = \nu_-(A)$. Now the stated equivalences can be obtained from Proposition 1.

This section is finished with an extension of Sylvester's law of inertia, which is actually obtained as a consequence of Theorem 1.

Proposition 2 *Let* $A = (A_{ij})_{i,j=1}^2$ *be an arbitrary selfadjoint block operator in* $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$, *which satisfies the range inclusion* (3), *and let* $S = |A_{11}|^{[-1/2]} A_{12}$. *Then* $\nu_-(A) < \infty$ *if and only if* $\nu_-(A_{11}) < \infty$ *and* $\nu_-(A_{22} - S^* J S) < \infty$; *in this case*

$$\nu_-(A) = \nu_-(A_{11}) + \nu_-(A_{22} - S^* J S).$$

In particular, $A \geq 0$ *if and only if* $\operatorname{ran} A_{12} \subset \operatorname{ran} |A_{11}|^{1/2}$, $A_{11} \geq 0$, *and* $A_{22} - S^* J S \geq 0$.

Proof By the assumption (3) $S = |A_{11}|^{[-1/2]} A_{12}$ is an everywhere defined bounded operator and, since $A_{11} = |A_{11}|^{1/2} J |A_{11}|^{1/2}$ (cf. Theorem 1), the following equality holds:

$$A = \begin{pmatrix} |A_{11}|^{1/2} & 0 \\ S^* J & I \end{pmatrix} \begin{pmatrix} J & 0 \\ 0 & A_{22} - S^* J S \end{pmatrix} \begin{pmatrix} |A_{11}|^{1/2} & J S \\ 0 & I \end{pmatrix},$$

i.e. $A = B^* E B$ where E stands for the diagonal operator with $\nu_-(E) = \nu_-(A_{11}) + \nu_-(A_{22} - S^* J S)$ and the triangular operator B on the right side is bounded and has dense range in $\overline{\operatorname{ran}}\, A_{11} \oplus \mathfrak{H}_2$. Clearly, $\nu_-(A) \leq \nu_-(E)$ and it remains to prove that if $\nu_-(A) < \infty$ then $\nu_-(A) = \nu_-(E)$.

To see this assume that $\nu_-(A) < \nu_-(E)$. We claim that ran B contains an E-negative subspace $\mathfrak{L}$ with dimension $\dim \mathfrak{L} > \nu_-(A)$. Assume the converse and let $\mathfrak{L} \subset \operatorname{ran} B$ be a maximal E-negative subspace with $\dim \mathfrak{L} \le \nu_-(A)$. Then $(E\mathfrak{L})^\perp$ must be E-nonnegative, since if $v \perp E\mathfrak{L}$ and $(Ev, v) < 0$, then span $\{v + \mathfrak{L}\}$ would be a proper E-negative extension of $\mathfrak{L}$. Since $E\mathfrak{L}$ is finite dimensional and ran B is dense in $\overline{\operatorname{ran}}\, A_{11} \oplus \mathfrak{H}_2$, ran B has dense intersection with $(\overline{\operatorname{ran}}\, A_{11} \oplus \mathfrak{H}_2) \ominus E\mathfrak{L}$, and hence the closure of this subspace is also E-nonnegative. Consequently, $\nu_-(E) = \nu_-(\mathfrak{L})$, a contradiction with the assumption $\nu_-(E) > \nu_-(A)$. This proves the claim that ran B contains an E-negative subspace $\mathfrak{L}$ with $\dim \mathfrak{L} > \nu_-(A)$. However, then the subspace $\mathfrak{L}' = \{u \in \overline{\operatorname{ran}}\, A_{11} \oplus \mathfrak{H}_2 : Bu \in \mathfrak{L}\}$ satisfies $\dim \mathfrak{L}' \ge \dim \mathfrak{L}$ and, moreover, $\mathfrak{L}'$ is A-negative: $(Au, u) = (EBu, Bu) < 0$, $u \in \mathfrak{L}'$, $u \ne 0$. Thus, $\nu_-(A) \ge \dim \mathfrak{L}$, a contradiction with $\dim \mathfrak{L} > \nu_-(A)$. This completes the proof.

Proposition 2 completes Theorem 1: if $\operatorname{ran} A_{12} \subset \operatorname{ran} |A_{11}|^{1/2}$ then $A_{11} = J|A_{11}|$ and $A_{12} = |A_{11}|^{1/2} S$ imply that $A_{21}|A_{11}|^{[-1/2]} J |A_{11}|^{[-1/2]} A_{12} = S^* J S$. Hence the negative index of A can be calculated by using the following version of a *generalized Schur complement* or a *shorted operator* (defined initially for a nonnegative operator H in (1))

$$A_{\mathfrak{H}_2} := \begin{pmatrix} 0 & 0 \\ 0 & A_{22} - S^* J S \end{pmatrix} \tag{17}$$

via the explicit formula

$$\nu_-(A) = \nu_-(A_{11}) + \nu_-(A_{22} - A_{21}|A_{11}|^{[-1/2]} J |A_{11}|^{[-1/2]} A_{12}). \tag{18}$$

The addition made in Proposition 2 concerns selfadjoint operators A_{22} that are not solutions to the original completion problem for A^0.

The notion of a shorted operator in infinite dimensional Hilbert spaces has been extended to the case of not necessarily selfadjoint block operators in a paper by Antezana et al. [6]. These so-called bilateral shorted operators introduced and studied therein use two range inclusions, see [6, Definitions 3.5, 4.1], which in the selfadjoint case reduce to the single condition (3) appearing in Theorems 1 and 2.

4 Lifting of operators with finite negative index

As a first application of the completion problem solved in Sect. 2 it is shown how nicely some lifting results established in a series of papers by Arsene, Constantinescu, and Gheondea, see [12,13,23,24], as well as in Dritschel and Rovnyak [33,34] (see also further references appearing in these papers) on contractive operators with finite number of negative squares can be derived from Theorem 1.

For this purpose some standard notations are introduced. Let $(\mathfrak{H}_1, (\cdot,\cdot)_1)$ and $(\mathfrak{H}_2, (\cdot,\cdot)_2)$ be Hilbert spaces and let J_1 and J_2 be symmetries in $\mathfrak{H}_1$ and $\mathfrak{H}_2$, i.e. $J_i = J_i^* = J_i^{-1}$, so that $(\mathfrak{H}_i, (J_i\cdot,\cdot)_i)$, $i = 1, 2$, becomes a Kreĭn space. Then associate with $T \in [\mathfrak{H}_1, \mathfrak{H}_2]$ the corresponding defect and signature operators

$$D_T = |J_1 - T^*J_2T|^{1/2}, \quad J_T = \operatorname{sign}(J_1 - T^*J_2T), \quad \mathfrak{D}_T = \overline{\operatorname{ran}}\, D_T,$$

where the so-called defect subspace $\mathfrak{D}_T$ can be considered as a Kreĭn space with the fundamental symmetry J_T. Similar notations are used with T^*:

$$D_{T^*} = |J_2 - TJ_1T^*|^{1/2}, \quad J_{T^*} = \operatorname{sign}(J_2 - TJ_1T^*), \quad \mathfrak{D}_{T^*} = \overline{\operatorname{ran}}\, D_{T^*}.$$

By definition $J_T D_T^2 = J_1 - T^*J_2T$ and $J_T D_T = D_T J_T$ with analogous identities for D_{T^*} and J_{T^*}. In addition,

$$\begin{aligned} (J_1 - T^*J_2T)J_1T^* &= T^*J_2(J_2 - TJ_1T^*), \\ (J_2 - TJ_1T^*)J_2T &= TJ_1(J_1 - T^*J_2T). \end{aligned} \tag{19}$$

Recall that $T \in [\mathfrak{H}_1, \mathfrak{H}_2]$ is said to be a J-contraction if $J_1 - T^*J_2T \geq 0$, i.e. $\nu_-(J_1 - T^*J_2T) = 0$. If, in addition, T^* is a J-contraction, T is termed as a J-bicontraction, in which case $\nu_-(J_1) = \nu_-(J_2)$ by Lemma 1. In what follows it is assumed that

$$\kappa_1 := \nu_-(J_1 - T^*J_2T) < \infty, \quad \kappa_2 := \nu_-(J_2 - TJ_1T^*) < \infty.$$

In this case Lemma 1 shows that

$$\nu_-(J_2) = \nu_-(J_1) + \kappa_2 - \kappa_1. \tag{20}$$

The aim in this section is to show applicability of Theorem 1 in establishing formulas for so-called liftings $\widetilde{T}$ of T with prescribed negative indices $\widetilde{\kappa}_1$ and $\widetilde{\kappa}_2$ for the defect subspaces, equivalently, for the associated signature operators. Given a bounded operator $T \in [\mathfrak{H}_1, \mathfrak{H}_2]$ the problem is to describe all operators $\widetilde{T}$ from the extended Kreĭn space $(\mathfrak{H}_1 \oplus \mathfrak{H}_1', J_1 \oplus J_1')$ to the extended Kreĭn space $(\mathfrak{H}_2 \oplus \mathfrak{H}_2', J_2 \oplus J_2')$ such that

$$(*) \quad P_2\widetilde{T}\upharpoonright\mathfrak{H}_1 = T \quad \text{and} \quad \nu_-(\widetilde{J}_1 - \widetilde{T}^*\widetilde{J}_2\widetilde{T}) = \widetilde{\kappa}_1, \quad \nu_-(\widetilde{J}_2 - \widetilde{T}\widetilde{J}_1\widetilde{T}^*) = \widetilde{\kappa}_2,$$

with some fixed values of $\widetilde{\kappa}_1, \widetilde{\kappa}_2 < \infty$. Here P_i stands for the orthogonal projection from $\widetilde{\mathfrak{H}}_i = \mathfrak{H}_i \oplus \mathfrak{H}_i'$ onto $\mathfrak{H}_i$ and $\widetilde{J}_i = J_i \oplus J_i'$, $i = 1, 2$. In addition, it is assumed that the exit spaces are Pontryagin spaces, i.e., that

$$\nu_-(J_1'), \nu_-(J_2') < \infty.$$

Following [13,23] consider first the following column extension problem:

$(*)_c$ Give a description of all operators $T_c = \operatorname{col}\begin{pmatrix} T & C \end{pmatrix} \in [\mathfrak{H}_1, \mathfrak{H}_2 \oplus \mathfrak{H}_2']$, such that $\nu_-(J_1 - T_c^*\widetilde{J}_2T_c) = \widetilde{\kappa}_1 \ (< \infty)$.

Since $J_1 - T_c^*\widetilde{J}_2T_c = J_1 - T^*J_2T - C^*J_2'C$, then necessarily (see Sect. 3)

$$\widetilde{\kappa}_1 \geq \kappa_1 - \nu_-(C^*J_2'C) \geq \kappa_1 - \nu_-(J_2').$$

Moreover, it is clear that $\widetilde{\kappa}_2 \geq \kappa_2$, since $J_2 - TJ_1T^*$ appears as the first diagonal entry of the 2×2 block operator $\widetilde{J}_2 - T_c J_1 T_c^*$ when decomposed w.r.t. $\widetilde{\mathfrak{H}}_i = \mathfrak{H}_i \oplus \mathfrak{H}_i'$, $i = 1, 2$.

With the minimal value of $\widetilde{\kappa}_1$ all solutions to this problem will now be described by applying Theorem 1 to an associated 2×2 block operator T_C appearing in the proof below; in fact the result is just a special case of Theorem 3.

Lemma 2 *Let* $\widetilde{\kappa}_1 = \nu_-(J_1 - T_c^* \widetilde{J}_2 T_c)$ *and assume that* $\widetilde{\kappa}_1 = \kappa_1 - \nu_-(J_2')(\geq 0)$. *Then* $\operatorname{ran} C^* \subset \operatorname{ran} D_T$ *and the formula*

$$T_c = \begin{pmatrix} T \\ K^* D_T \end{pmatrix}$$

establishes a one-to-one correspondence between the set of all solutions to Problem $(*)_c$ *and the set of all* J*-contractions* $K \in [\mathfrak{H}_2', \mathfrak{D}_T]$.

Proof To make the argument more explicit consider the following block operator

$$T_C := \begin{pmatrix} J_1 - T^* J_2 T & C^* \\ C & J_2' \end{pmatrix} = \begin{pmatrix} I & C^* J_2' \\ 0 & I \end{pmatrix} \begin{pmatrix} J_1 - T_c^* \widetilde{J}_2 T_c & 0 \\ 0 & J_2' \end{pmatrix} \begin{pmatrix} I & 0 \\ J_2' C & I \end{pmatrix}.$$

Clearly $\nu_-(T_C) = \nu_-(J_1 - T_c^* \widetilde{J}_2 T_c) + \nu_-(J_2') < \infty$, which combined with $\widetilde{\kappa}_1 = \kappa_1 - \nu_-(J_2')$ shows that $\nu_-(T_C) = \kappa_1 = \nu_-(J_1 - T^* J_2 T)$. Now, the statement is obtained from Theorem 1 or, more directly, just by applying Theorem 3.

Remark 1 (i) The above proof, which essentially makes use of an associated 2×2 block operator T_C (being a special case of the block operator H in (12) behind Theorem 3), is new even in the case of Hilbert space contractions. In particular, it shows that the operator K in Lemma 2 coincides with the operator S that gives the minimal solution $S^* J_T S$ to the completion problem associated with T_C; the J-contractivity of K itself is equivalent to the fact that T_C is also a solution precisely when $\widetilde{\kappa} = \kappa - \nu_-(J_2')$.

(ii) The existence of a solution to Problem $(*)_c$ is proved here using only the condition $\widetilde{\kappa}_1 = \kappa_1 - \nu_-(J_2')\,(\geq 0)$. The corresponding result in [23, Lemma 2.2] is formulated (and formally also proved) under the additional condition $\widetilde{\kappa}_2 = \kappa_2$. In the case that $\nu_-(J_1) < \infty$ the equality $\widetilde{\kappa}_2 = \kappa_2$ follows automatically from the equality $\widetilde{\kappa}_1 = \kappa_1 - \nu_-(J_2')$: to see this apply (20) to T and T_c, which leads to $\nu_-(J_1) + \kappa_2 = \nu_-(J_1) + \widetilde{\kappa}_2$, so that $\nu_-(J_1) < \infty$ implies $\kappa_2 = \widetilde{\kappa}_2$. Naturally, in Lemma 2 the condition $\widetilde{\kappa}_2 = \kappa_2$ follows from the condition $\widetilde{\kappa}_1 = \kappa_1 - \nu_-(J_2')$ also in the case where $\nu_-(J_1) = \infty$; see Corollary 3 below.

Finally, it is mentioned that for a Pontryagin space operator T the result in Lemma 2 was proved in [13, Lemma 5.2].

In a dual manner we can treat the following row extension problem; again initially considered in [13,23]:

$(*)_\mathbf{r}$ Give a description of all operators $T_r = \begin{pmatrix} T & R \end{pmatrix} \in [\mathfrak{H}_1 \oplus \mathfrak{H}_1', \mathfrak{H}_2]$, such that $\nu_-(J_2 - T_r \widetilde{J}_1 T_r^*) = \widetilde{\kappa}_2\ (< \infty)$.

Analogous to the case of column operators, $J_2 - T_r\widetilde{J}_1T_r^* = J_2 - TJ_1T^* - RJ_1'R^*$ gives the estimate

$$\widetilde{\kappa}_2 \geq \kappa_2 - \nu_-(RJ_1'R^*) \geq \kappa_2 - \nu_-(J_1').$$

Moreover, it is clear that $\widetilde{\kappa}_1 \geq \kappa_1$. With the minimal value of $\widetilde{\kappa}_2$ all solutions to Problem $(*)_\mathbf{r}$ are established by applying Theorem 1 to an associated 2×2 block operator T_R.

Lemma 3 *Let* $\widetilde{\kappa}_2 = \nu_-(J_2 - T_r\widetilde{J}_1T_r^*)$ *and assume that* $\widetilde{\kappa}_2 = \kappa_2 - \nu_-(J_1')(\geq 0)$. *Then* $\operatorname{ran} R \subset \operatorname{ran} D_{T^*}$ *and the formula*

$$T_r = \begin{pmatrix} T & D_{T^*}B \end{pmatrix}$$

establishes a one-to-one correspondence between the set of all solutions to Problem $(*)_\mathbf{r}$ *and the set of all* J*-contractions* $B \in [\mathfrak{H}_1', \mathfrak{D}_{T^*}]$.

Proof To prove the statement via Theorem 1 (cf. Theorem 3) consider

$$T_R := \begin{pmatrix} J_2 - TJ_1T^* & R \\ R^* & J_1' \end{pmatrix} = \begin{pmatrix} I & RJ_1' \\ 0 & I \end{pmatrix} \begin{pmatrix} J_2 - T_r\widetilde{J}_1T_r^* & 0 \\ 0 & J_1' \end{pmatrix} \begin{pmatrix} I & 0 \\ J_1'R^* & I \end{pmatrix}.$$

Then clearly $\nu_-(T_R) = \nu_-(J_2 - T_r\widetilde{J}_1T_r^*) + \nu_-(J_1')$ and hence the assumption $\widetilde{\kappa}_2 = \kappa_2 - \nu_-(J_1')$ is equivalent to $\nu_-(T_R) = \kappa_2 = \nu_-(J_2 - TJ_1T^*)$. Therefore, again the statement follows from Theorem 1 or directly from Theorem 3.

Remarks similar to those made after Lemma 2 can be done here, too. In particular, the corresponding result in [23, Lemma 2.1] is formulated under the additional condition $\widetilde{\kappa}_1 = \kappa_1$: here this equality will be a consequence from the equality $\widetilde{\kappa}_2 = \kappa_2 - \nu_-(J_1')$; cf. Corollary 3 below.

To prove the main result concerning parametrization of all 2×2 liftings in a larger Kreĭn space with minimal signature for the defect operators an indefinite version of the commutation relation of the form $TD_T = D_{T^*}T$ is needed; these involve so-called link operators introduced in [13, Section 4].

We will give a simple proof for the construction of link operators (see [13, Proposition 4.1]) by applying Heinz inequality combined with the basic factorization result from [32]. The first step is formulated in the next lemma, which is connected to a result of Kreĭn [48] concerning continuity of a bounded Banach space operator which is symmetric w.r.t. to a continuous definite inner product; the existence of link operators was proved in [13] via this result of Kreĭn. Here a statement, analogous to that of Kreĭn, is formulated in pure Hilbert space operator language by using the modulus of the product operator; see [34, Lemma B2], where Kreĭn's result is presented with a proof due to W. T. Reid.

Lemma 4 *Let* $S \in [\mathfrak{H}_1, \mathfrak{H}_2]$ *and let* $H \in [\mathfrak{H}_2]$ *be nonnegative. Then*

$$HS = (HS)^* \quad \Rightarrow \quad |HS| \leq \mu H \text{ for some } \mu < \infty.$$

Proof Since HS is selfadjoint, one obtains

$$(HS)^2 = HSS^*H \leq \mu^2 H^2, \quad \mu = \|S\| < \infty.$$

Now by Heinz inequality (see e.g. [17, Theorem 10.4.2]) we get

$$|HS| = (HSS^*H)^{1/2} \leq \mu H.$$

Corollary 2 *Let $T \in [\mathfrak{H}_1, \mathfrak{H}_2]$ and let J_1 and J_2 be symmetries in $\mathfrak{H}_1$ and $\mathfrak{H}_2$ as above. Then there exist unique operators $L_T \in [\mathfrak{D}_T, \mathfrak{D}_{T^*}]$ and $L_{T^*} \in [\mathfrak{D}_{T^*}, \mathfrak{D}_T]$ such that*

$$D_{T^*} L_T = T J_1 D_T \upharpoonright \mathfrak{D}_T, \quad D_T L_{T^*} = T^* J_2 D_{T^*} \upharpoonright \mathfrak{D}_{T^*};$$

in fact, $L_T = D_{T^*}^{[-1]} T J_1 D_T \upharpoonright \mathfrak{D}_T$ and $L_{T^*} = D_T^{[-1]} T^* J_2 D_{T^*} \upharpoonright \mathfrak{D}_{T^*}$.

Proof Denote $S = J_{T^*} J_2 T J_T J_1 T^*$. Then (19) implies that

$$\begin{aligned} D_{T^*}^2 S &= (J_2 - T J_1 T^*) J_2 T J_T J_1 T^* \\ &= T J_1 (J_1 - T^* J_2 T) J_T J_1 T^* \\ &= T J_1 D_T^2 J_1 T^* \geq 0, \end{aligned}$$

so that $D_{T^*}^2 S$ is nonnegative and, in particular, selfadjoint. By Lemma 4 with $\mu = \|S\|$ one has

$$0 \leq T J_1 D_T^2 J_1 T^* = D_{T^*}^2 S \leq \mu D_{T^*}^2.$$

This last inequality is equivalent to the factorization $T J_1 D_T \upharpoonright \mathfrak{D}_T = D_{T^*} L_T$ with a unique operator $L_T \in [\mathfrak{D}_T, \mathfrak{D}_{T^*}]$, see [32, Theorem 1], which by means of Moore–Penrose generalized inverse can be rewritten as indicated.

The second formula is obtained by applying the first one to T^*.

The following identities can be obtained with direct calculations; see [13, Section 4]:

$$\begin{aligned} L_T^* J_{T^*} \upharpoonright \mathfrak{D}_{T^*} &= J_T L_{T^*}; \\ (J_T - D_T J_1 D_T) \upharpoonright \mathfrak{D}_T &= L_T^* J_{T^*} L_T; \\ (J_{T^*} - D_{T^*} J_2 D_{T^*}) \upharpoonright \mathfrak{D}_{T^*} &= L_{T^*}^* J_T L_{T^*}. \end{aligned} \tag{21}$$

The next corollary contains the promised identity $\widetilde{\kappa}_1 = \kappa_1$ under the assumption $\widetilde{\kappa}_2 = \kappa_2 - \nu_-(J_2') \geq 0$ in Lemma 3. Similarly $\widetilde{\kappa}_1 = \kappa_1 - \nu_-(J_1')$ implies $\widetilde{\kappa}_2 = \kappa_2$; the general result for the first case can be formulated as follows (and there is similar result for the latter case).

Corollary 3 *Let R be a bounded operator such that* ran $R \subset$ ran D_{T^*} *and let T_r be the corresponding row operator and denote $\widetilde{\kappa}_1 = \nu_-(\widetilde{J}_1 - T_r^* J_2 T_r)$. Then $R = D_{T^*} B$ for a (unique) bounded operator $B \in [\mathfrak{H}_1', \mathfrak{D}_{T^*}]$ and*

$$\widetilde{\kappa}_1 = \kappa_1 + \nu_-(J_1' - B^* J_{T^*} B).$$

In particular, J-contractivity of B is equivalent to $\widetilde{\kappa}_1 = \kappa_1$.

Proof Recall that ran $R \subset$ ran D_{T^*} is equivalent to the factorization $R = D_{T^*}B$. By applying the commutation relations in Corollary 2 together with the identities (21) one gets the following expression for $J_{T_r}D^2_{T_r}$:

$$J_{T_r}D^2_{T_r} = \begin{pmatrix} J_1 - T^*J_2T & -T^*J_2D_{T^*}B \\ -B^*D_{T^*}J_2T & J_1' - B^*D_{T^*}J_2D_{T^*}B \end{pmatrix} = \begin{pmatrix} J_TD^2_T & -D_TL_{T^*}B \\ -B^*L^*_{T^*}D_T & J_BD^2_B + B^*L^*_{T^*}J_TL_{T^*}B \end{pmatrix}. \tag{22}$$

Now apply Proposition 2 and calculate the Schur complement, cf. (18),

$$J_BD^2_B + B^*L^*_{T^*}J_TL_{T^*}B - B^*L^*_{T^*}D_T(D_T^{[-1]}J_TD_T^{[-1]})D_TL_{T^*}B = J_BD^2_B,$$

to see that $\widetilde{\kappa}_1 = \nu_-(J_1 - T^*J_2T) + \nu_-(J_1' - B^*J_{T^*}B)$.

By means of Lemmas 2, 3 and the link operators in Corollary 2 one can now establish the main result concerning the lifting problem $(*)$.

First notice that if Problem $(*)$ has a solution, then by treating $\widetilde{T}$ as a row extension of its first column T_c and as a column extension of its first row T_r one gets from the inequalities preceding Lemmas 2, 3 the estimates

$$\begin{aligned} \widetilde{\kappa}_1 \geq \kappa_1(T_r) - \nu_-(J_2') \geq \kappa_1 - \nu_-(J_2'); \\ \widetilde{\kappa}_2 \geq \kappa_2(T_c) - \nu_-(J_1') \geq \kappa_2 - \nu_-(J_1'). \end{aligned} \tag{23}$$

Under the minimal choice of the indices $\widetilde{\kappa}_1$ and $\widetilde{\kappa}_2$ Problem $(*)$ is already solvable; all solutions are described by the following result, which was initially proved in [23, Theorem 2.3] with the aid of [13, Theorem 5.3]. Here a different proof is presented, again based on an application of Theorem 1.

Theorem 4 *Let $\widetilde{T}$ be a bounded operator from $(\mathfrak{H}_1 \oplus \mathfrak{H}_1', J_1 \oplus J_1')$ to $(\mathfrak{H}_2 \oplus \mathfrak{H}_2', J_2 \oplus J_2')$ such that $P_2\widetilde{T}\upharpoonright\mathfrak{H}_1 = T$. Assume that $0 \leq \kappa_1 - \nu_-(J_2') = \widetilde{\kappa}_1 < \infty$ and $0 \leq \kappa_2 - \nu_-(J_1') = \widetilde{\kappa}_2 < \infty$. Then the Problem $(*)$ is solvable and the formula*

$$\widetilde{T} = \begin{pmatrix} T & D_{T^*}\Gamma_1 \\ \Gamma_2D_T & -\Gamma_2L^*_TJ_{T^*}\Gamma_1 + D_{\Gamma_2^*}\Gamma D_{\Gamma_1} \end{pmatrix}$$

establishes a one-to-one correspondence between the set of all solutions to Problem $(*)$ and the set of triplets $\{\Gamma_1, \Gamma_2, \Gamma\}$ where $\Gamma_1 \in [\mathfrak{H}_1', \mathfrak{D}_{T^*}]$ and $\Gamma_2^* \in [\mathfrak{H}_2', \mathfrak{D}_T]$ are J-contractions and $\Gamma \in [\mathfrak{D}_{\Gamma_1}, \mathfrak{D}_{\Gamma_2^*}]$ is a Hilbert space contraction.

Proof Assume that there is a solution $\widetilde{T}$ to Problem $(*)$ and write it in the form

$$\widetilde{T} = \begin{pmatrix} T & R \\ C & X \end{pmatrix}$$

with the first column denoted by T_c and first row denoted by T_r, and assume that $\widetilde{\kappa}_1 = \kappa_1 - \nu_-(J_2')$ and $\widetilde{\kappa}_2 = \kappa_2 - \nu_-(J_1')$. Then (23) shows that $\kappa_1 = \kappa_1(T_r)$ and

$\kappa_2 = \kappa_2(T_c)$. Hence Lemma 3 can be applied by viewing $\widetilde{T}$ as a row extension of T_c to get a range inclusion and then from Corollary 3 one gets the equality $\widetilde{\kappa}_1 = \kappa_1(T_c)$. Similarly applying Lemma 2 and the analog of Corollary 3 to column operator $\widetilde{T}$ one gets the equality $\widetilde{\kappa}_2 = \kappa_2(T_r)$. Thus $\kappa_1(T_c) = \kappa_1 - \nu_-(J_2')$ and $\kappa_2(T_r) = \kappa_2 - \nu_-(J_1')$. Consequently, one can apply Lemma 2 to the first column T_c and Lemma 3 to the first row T_r to get the stated factorizations $C = \Gamma_2 D_T$ and $R = D_{T^*}\Gamma_1$ with unique J-contractions Γ_1 and Γ_2^*.

To establish a formula for X we proceed by considering the block operator

$$H := \begin{pmatrix} J_{T_r} D_{T_r}^2 & T_{r,2}^* \\ T_{r,2} & J_2' \end{pmatrix},$$

where $T_{r,2}$ denotes the second row of $\widetilde{T}$. It is straightforward to derive the following formula for the Schur complement

$$J_{T_r} D_{T_r}^2 - T_{r,2}^* J_2' T_{r,2} = \widetilde{J}_1 - \widetilde{T}^* \widetilde{J}_2 \widetilde{T}.$$

Thus $\nu_-(H) = \widetilde{\kappa}_1 + \nu_-(J_2') = \kappa_1 = \nu_-(J_{T_r})$ and one can apply Theorem 1 to get the factorization $T_{r,2}^* = D_{T_r}\widetilde{K}$ with a unique $\widetilde{K} \in [\mathfrak{H}_2', \mathfrak{D}_{T_r}]$ satisfying $\widetilde{K}^* J_{T_r} \widetilde{K} \le J_2'$, i.e., $\widetilde{K}$ is a J-contraction; see Theorem 3.

It follows from (22) that

$$J_{T_r} D_{T_r}^2 = \begin{pmatrix} D_T & 0 \\ -\Gamma_1^* L_{T^*}^* J_T & D_{\Gamma_1} \end{pmatrix} \begin{pmatrix} J_T & 0 \\ 0 & I_{D_{\Gamma_1}} \end{pmatrix} \begin{pmatrix} D_T & -J_T L_{T^*}\Gamma_1 \\ 0 & D_{\Gamma_1} \end{pmatrix} =: B^* \widehat{J} B.$$

Since here $\nu_-(J_{T_r}) = \kappa_1 = \nu_-(J_T)$ and B is a triangular operator whose range is dense in $\mathfrak{D}_T \oplus \mathfrak{D}_{\Gamma_1}$ (the diagonal entries D_T and D_{Γ_1} of B have dense ranges by definition), there is a unique Pontryagin space J-unitary operator U from $\mathfrak{D}_{T_r}$ onto $\mathfrak{D}_T \oplus \mathfrak{D}_{\Gamma_1}$ such that $B = U D_{T_r}$; see Proposition 1 (ii). It follows that $K^* := (U^{-1})^* \widetilde{K}$ is a J-contraction from $\mathfrak{H}_2'$ to $\mathfrak{D}_T \oplus \mathfrak{D}_{\Gamma_1}$ and $KB = \widetilde{K}^* D_{T_r} = T_{r,2}$. Now $J_2' - K\widehat{J}K^* \ge 0$ gives

$$0 \le K_1 K_1^* \le J_2' - K_0 J_T K_0^*, \tag{24}$$

where $K = (K_0\ K_1)$ is considered as a row operator, and $T_{r,2} = KB$ reads as

$$\Gamma_2 D_T = K_0 D_T, \quad X = -K_0 J_T L_{T^*}\Gamma_1 + K_1 D_{\Gamma_1}.$$

Since all contractions that are involved are unique, $K_0 = \Gamma_2$, $J_2' - K_0 J_T K_0^* = D_{\Gamma_2^*}^2$, and (24) implies that there is a unique Hilbert space contraction $\Gamma \in [\mathfrak{D}_{\Gamma_1}, \mathfrak{D}_{\Gamma_2^*}]$ such that $K_1 = D_{\Gamma_2^*}\Gamma$. The desired formula for $\widetilde{T}$ is proven (cf. (21)). It is clear from the proof that every operator $\widetilde{T}$ of the stated form is a solution and that there is one-to-one correspondence via the triplets $\{\Gamma_1, \Gamma_2, \Gamma\}$ of J-contractions.

Remark 2 (i) By replacing $\widetilde{T}$ with its adjoint $\widetilde{T}^*$ it is clear that all formulas remain the same and are obtained by changing T with T^* and interchanging the roles of the indices 1 and 2; see also (21). This connects the considerations with row and column operators to each other.

(ii) If $\kappa_1 = 0$ so that $J_1 - T^* J_2 T \geq 0$, then the above proof becomes slightly simpler since then J_{T_r}, J_T, and J_2' are identity operators and $\widetilde{K}$ is a Hilbert space contraction. Then Theorem 4 gives all contractive liftings of a contraction in a Kreĭn space. If in addition $\kappa_2 = 0$, then one gets all bicontractive liftings of a bicontraction in a Kreĭn space with Pontryagin spaces as exit spaces. In the special case that the exit spaces are Hilbert spaces ($\nu_-(J_1) = \nu_-(J_2) = 0$ and $\kappa_1 = \kappa_2 = 0$) Theorem 4 coincides with [33, Theorem 3.6]. In fact, the present proof can be seen as a further development of the proof appearing in that paper; see also further references and historical remarks given in [33,34].

5 Contractive extensions of contractions with minimal negative indices

Let $\mathfrak{H}_1$ be a closed linear subspace of the Hilbert space $\mathfrak{H}$, let $T_{11} = T_{11}^* \in [\mathfrak{H}_1]$ be an operator such that $\nu_-(I - T_{11}^2) = \kappa < \infty$. Denote

$$J = \operatorname{sign}(I - T_{11}^2), \quad J_+ = \operatorname{sign}(I - T_{11}), \text{ and } J_- = \operatorname{sign}(I + T_{11}), \tag{25}$$

and let $\kappa_+ = \nu_-(I - T_{11})$ and $\kappa_- = \nu_-(I + T_{11})$. It is obvious that $J = J_- J_+ = J_+ J_-$. Moreover, there is an equality $\kappa = \kappa_- + \kappa_+$ as stated in the next lemma.

Lemma 5 *Let $T = T^* \in [\mathfrak{H}_1]$ be an operator such that $\nu_-(I - T^2) = \kappa < \infty$ then $\nu_-(I - T^2) = \nu_-(I + T) + \nu_-(I - T)$.*

Proof Let $E_t(\cdot)$ be the resolution of identity of T. Then by the spectral mapping theorem the spectral subspace corresponding to the negative spectrum of $I - T^2$ is given by $E_t((\infty; -1) \cup (1; \infty)) = E_t((-\infty; -1)) \oplus E_t((1; \infty))$. Consequently, $\nu_-(I - T^2) = \dim E_t((-\infty; -1)) + \dim E_t((1; \infty)) = \nu_-(I + T) + \nu_-(I - T)$.

The next problem concerns the existence and a description of selfadjoint operators T such that $\widetilde{A}_+ = I + T$ and $\widetilde{A}_- = I - T$ solve the corresponding completion problems

$$A_\pm^0 = \begin{pmatrix} I \pm T_{11} & \pm T_{21}^* \\ \pm T_{21} & * \end{pmatrix}, \tag{26}$$

under *minimal index conditions* $\nu_-(I + T) = \nu_-(I + T_{11})$, $\nu_-(I - T) = \nu_-(I - T_{11})$, respectively. Observe, that if $I \pm T$ provides an arbitrary completion to $A_\pm^0$ then clearly $\nu_-(I \pm T) \geq \nu_-(I \pm T_{11})$. Thus by Lemma 5 the two minimal index conditions above are equivalent to the single condition $\nu_-(I - T^2) = \nu_-(I - T_{11}^2)$.

Unlike with the case of a selfadjoint contraction T_{11}, this problem need not have solutions when $\nu_-(I - T_{11}^2) > 0$. It is clear from Theorem 1 that the conditions $\operatorname{ran} T_{21}^* \subset \operatorname{ran} |I - T_{11}|^{1/2}$ and $\operatorname{ran} T_{21}^* \subset \operatorname{ran} |I + T_{11}|^{1/2}$ are necessary for the existence of solutions; however alone they are not sufficient.

The next theorem gives a general solvability criterion for the completion problem (26) and describes all solutions to this problem, when the criterion is satisfied. As in the definite case, there are minimal solutions A_+ and A_- which are connected to two extreme selfadjoint extensions T of

$$T_1 = \begin{pmatrix} T_{11} \\ T_{21} \end{pmatrix} : \mathfrak{H}_1 \to \begin{pmatrix} \mathfrak{H}_1 \\ \mathfrak{H}_2 \end{pmatrix}, \tag{27}$$

now with finite negative index $\nu_-(I-T^2) = \nu_-(I-T_{11}^2) > 0$. The set of all solutions T to the problem (26) will be denoted by $\mathrm{Ext}_{T_1,\kappa}(-1,1)$.

Theorem 5 *Let T_1 be a symmetric operator as in (27) with $T_{11} = T_{11}^* \in [\mathfrak{H}_1]$ and $\nu_-(I - T_{11}^2) = \kappa < \infty$, and let $J = \mathrm{sign}\,(I - T_{11}^2)$. Then the completion problem for $A_\pm^0$ in (26) has a solution $I \pm T$ for some $T = T^*$ with $\nu_-(I - T^2) = \kappa$ if and only if the following condition is satisfied*:

$$\nu_-(I - T_{11}^2) = \nu_-(I - T_1^* T_1). \tag{28}$$

If this condition is satisfied then the following facts hold:

(i) *The completion problems for $A_\pm^0$ in (26) have minimal solutions $A_\pm$.*
(ii) *The operators $T_m := A_+ - I$ and $T_M := I - A_- \in \mathrm{Ext}_{T_1,\kappa}(-1,1)$.*
(iii) *The operators T_m and T_M have the block forms*

$$\begin{aligned} T_m &= \begin{pmatrix} T_{11} & D_{T_{11}} V^* \\ V D_{T_{11}} & -I + V(I - T_{11}) J V^* \end{pmatrix}, \\ T_M &= \begin{pmatrix} T_{11} & D_{T_{11}} V^* \\ V D_{T_{11}} & I - V(I + T_{11}) J V^* \end{pmatrix}, \end{aligned} \tag{29}$$

where $D_{T_{11}} := |I - T_{11}^2|^{1/2}$ and V is given by $V := \mathrm{clos}\,(T_{21} D_{T_{11}}^{[-1]})$.
(iv) *The operators T_m and T_M are extremal extensions of T_1:*

$$T \in \mathrm{Ext}_{T_1,\kappa}(-1,1) \;\; \textit{iff} \;\; T = T^* \in [\mathfrak{H}], \quad T_m \le T \le T_M. \tag{30}$$

(v) *The operators T_m and T_M are connected via*

$$(-T)_m = -T_M, \quad (-T)_M = -T_m. \tag{31}$$

Proof It is easy to see that $\kappa = \nu_-(I - T_{11}^2) \le \nu_-(I - T_1^* T_1) \le \nu_-(I - T^2)$. Hence the condition $\nu_-(I - T^2) = \kappa$ implies (28). The sufficiency of this condition is established while proving the assertions (i)–(iii) below. (i) If the condition (28) is satisfied then $\mathrm{ran}\, T_{21}^* \subset \mathrm{ran}\,|I - T_{11}^2|^{1/2}$ by Lemma 2. In fact, this inclusion is equivalent to the inclusions $\mathrm{ran}\, T_{21}^* \subset \mathrm{ran}\,|I \pm T_{11}|^{1/2}$, which by Theorem 1 means that both of the completion problems, $A_\pm^0$ in (26), are solvable. Consequently, the following operators

$$S_- = |I + T_{11}|^{[-1/2]} T_{21}^*, \quad S_+ = |I - T_{11}|^{[-1/2]} T_{21}^* \tag{32}$$

are well defined and they provide the minimal solutions $A_\pm$ to the completion problems for $A_\pm^0$ in (26). Notice that the assumption that there is a simultaneous solution $I \pm T$ with a *single* selfadjoint operator T is not yet used here.

(ii) & (iii) Proof of (i) shows that the inclusion $\operatorname{ran} T_{21}^* \subset \operatorname{ran} |I - T_{11}^2|^{1/2}$ holds. This last inclusion alone is equivalent to the existence of a (unique) bounded operator $V^* = D_{T_{11}}^{[-1]} T_{21}^*$ with $\ker V \supset \ker D_{T_{11}}$, such that $T_{21}^* = D_{T_{11}} V^*$. The operators $T_m := A_+ - I$ and $T_M := I - A_-$ (see proof of (i)) can be now rewritten as in (29). Observe that

$$S_\mp = |I \pm T_{11}|^{[-1/2]} D_{T_{11}} V^* = P_\mp |I \mp T_{11}|^{1/2} V^* = |I \mp T_{11}|^{1/2} P_\mp V^*,$$

where $P_\mp$ are the orthogonal projections onto

$$(\ker |I \pm T_{11}|^{1/2})^\perp = (\ker |I \pm T_{11}|)^\perp = \overline{\operatorname{ran}} |I \pm T_{11}| = \overline{\operatorname{ran}} |I \pm T_{11}|^{1/2}.$$

Since $\ker V \supset \ker D_{T_{11}}$ implies $\overline{\operatorname{ran}}\, V^* \subset \overline{\operatorname{ran}}\, D_{T_{11}} \subset \overline{\operatorname{ran}} |I \pm T_{11}|^{1/2}$, it follows that

$$S_- = |I - T_{11}|^{1/2} V^*, \quad S_+ = |I + T_{11}|^{1/2} V^*.$$

Consequently, see (25),

$$\begin{aligned} S_-^* J_- S_- &= V |I - T_{11}|^{1/2} J_- |I - T_{11}|^{1/2} V^* = V(I - T_{11}) J V^*, \\ S_+^* J_+ S_+ &= V |I + T_{11}|^{1/2} J_+ |I + T_{11}|^{1/2} V^* = V(I + T_{11}) J V^*, \end{aligned}$$

which implies the representations for T_m and T_M in (29). Clearly, T_m and T_M are selfadjoint extensions of T_1, which satisfy the equalities

$$\nu_-(I + T_m) = \kappa_-, \quad \nu_-(I - T_M) = \kappa_+.$$

Moreover, it follows from (29) that

$$T_M - T_m = \begin{pmatrix} 0 & 0 \\ 0 & 2(I - VJV^*) \end{pmatrix}. \tag{33}$$

Now the assumption (28) will be used again. Since $\nu_-(I - T_1^* T_1) = \nu_-(I - T_{11}^2)$ and $T_{21} = V D_{T_{11}}$ it follows from Lemma 2 that $V^* \in [\mathfrak{H}_2, \mathfrak{D}_{T_{11}}]$ is J-contractive: $I - VJV^* \geq 0$. Therefore, (33) shows that $T_M \geq T_m$ and $I + T_M \geq I + T_m$ and hence, in addition to $I + T_m$, also $I + T_M$ is a solution to the problem A_+^0 and, in particular, $\nu_-(I + T_M) = \kappa_- = \nu_-(I + T_m)$. Similarly, $I - T_M \leq I - T_m$ which implies that $I - T_m$ is also a solution to the problem A_-^0, in particular, $\nu_-(I - T_m) = \kappa_+ = \nu_-(I - T_M)$. Now by applying Lemma 5 we get

$$\begin{aligned} \nu_-(I - T_m^2) &= \nu_-(I - T_m) + \nu_-(I + T_m) = \kappa_+ + \kappa_- = \kappa, \\ \nu_-(I - T_M^2) &= \nu_-(I - T_M) + \nu_-(I + T_M) = \kappa_+ + \kappa_- = \kappa. \end{aligned}$$

Therefore, $T_m, T_M \in \operatorname{Ext}_{T_1,\kappa}(-1,1)$ which in particular proves that the condition (28) is sufficient for solvability of the completion problem (26).

(iv) Observe, that $T \in \operatorname{Ext}_{T_1,\kappa}(-1,1)$ if and only if $T = T^* \supset T_1$ and $\nu_-(I \pm T) = \kappa_\mp$. By Theorem 1 this is equivalent to

$$S_-^* J_- S_- - I \le T_{22} \le I - S_+^* J_+ S_+. \tag{34}$$

The inequalities (34) are equivalent to (30).

(v) The relations (31) follow from (32) and (29).

For a Hilbert space contraction T_1 one has $\nu_-(I - T_{11}^2) \le \nu_-(I - T_1^* T_1) = 0$, i.e., the criterion (28) is automatically satisfied. In this case Theorem 5 has been proved in [39]. As Theorem 5 shows, under the minimal index condition $\nu_-(I - T^2) = \nu_-(I - T_{11}^2)$, the solution set $\operatorname{Ext}_{T_1,\kappa}(-1,1)$ admits the same attractive description as an operator interval determined by the two extreme extensions T_m and T_M as was originally proved by Kreĭn in his paper [47] when describing all contractive selfadjoint extensions of a Hilbert space contraction. In particular, Theorem 5 shows that if there is a solution to the completion problem (26), i.e. if T_1 satisfies the index condition (28), then all selfadjoint extensions T of T_1 satisfying the equality $\nu_-(I - T^2) = \nu_-(I - T_1^* T_1)$ are determined by the operator inequalities $T_m \le T \le T_M$. The original paper [47] of M. G. Kreĭn has never been translated: for some literature in English where many of the original ideas of Kreĭn have been presented we refer to the monographs [1,9,51] and the papers [11,39].

The original proof of Kreĭn in [47] for the description of all contractive selfadjoint extensions of a Hilbert space contraction T_1 as the operator interval in (30) was based on the notion of shortening or shorted operator; cf. (1). To get this result Kreĭn first constructed one contractive selfadjoint extension T for T_1 and then used it together with the following two formulas involving shortening of $I + T$ and $I - T$ to the subspace $\mathfrak{N} = \mathfrak{H} \ominus \operatorname{dom} T_1 = \mathfrak{H}_2$:

$$T_m = T - (I + T)_{\mathfrak{N}}, \quad T_M = T + (I - T)_{\mathfrak{N}},$$

see [47, Theorem 3]. It follows from Theorem 1, see also (10), and the formulas for T_m and T_M in Theorem 5 that these descriptions of T_m and T_M remain true in the present setting: indeed, using the given block formulas one can directly check that

$$I + T = I + T_m + (I + T)_{\mathfrak{N}}, \quad I - T = I - T_M + (I - T)_{\mathfrak{N}},$$

where the shortening is calculated as defined in (17).

Notice that T belongs to the solution set $\operatorname{Ext}_{T_1,\kappa}(-1,1)$ precisely when $T = T^* \supset T_1$ and $\nu_-(I \pm T) = \kappa_\mp$. This means that every selfadjoint extension of T_1 for which $(I - T^2) = \nu_-(I - T_1^* T_1)$ admits precisely κ_- eigenvalues on the interval $(-\infty, -1)$ and κ_+ eigenvalues on the interval $(1, \infty)$; in total there are $\kappa = \kappa_- + \kappa_+$ eigenvalues outside the closed interval $[-1, 1]$. The fact that the numbers $\kappa_\mp = \nu_-(I \pm T)$ are constant in the solution set $\operatorname{Ext}_{T_1,\kappa}(-1,1)$ is crucial for dealing properly with the Cayley transforms in the next section.

6 A generalization of M. G. Kreĭn's approach to the extension theory of nonnegative operators

6.1 Some antitonicity theorems for selfadjoint relations

The notion of inertia of a selfadjoint relation in a Hilbert space is defined by means of its associated spectral measure. In what follows the Hilbert space is assumed to be separable.

Definition 1 Let H be a selfadjoint relation in a separable Hilbert space $\mathfrak{H}$ and let $E_t(\cdot)$ be the spectral measure of H. The inertia of H is defined as the ordered quadruplet $\mathsf{i}(H) = \{\mathsf{i}^+(H), \mathsf{i}^-(H), \mathsf{i}^0(H), \mathsf{i}^\infty(H)\}$, where

$$\mathsf{i}^+(H) = \dim \operatorname{ran} E_t((0,\infty)), \quad \mathsf{i}^-(H) = \dim \operatorname{ran} E_t((-\infty,0)),$$
$$\mathsf{i}^0(H) = \dim \ker H, \quad \mathsf{i}^\infty(H) = \dim \operatorname{mul} H.$$

In particular, for a selfadjoint relation H in $\mathbb{C}^n$, the quadruplet $\mathsf{i}(H)$ consists of the numbers of positive, negative, zero, and infinite eigenvalues of H; cf. [15]. Hence, if H is a selfadjoint matrix in $\mathbb{C}^n$, then $\mathsf{i}^\infty(H) = 0$ and the remaining numbers make up the usual inertia of H.

The following theorem characterizes the validity of the implication

$$H_1 \le H_2 \quad \Rightarrow \quad H_2^{-1} \le H_1^{-1}$$

for a pair of bounded selfadjoint operators H_1 and H_2 having bounded inverses; in the infinite dimensional case it has been proved independently in [30,40,61]; cf. also [41]. Some extensions of this result, where the condition $\min\{\mathsf{i}_2^+, \mathsf{i}_1^-\} < \infty$ is relaxed, are also contained in [40,41,61].

Theorem 6 *Let H_1 and H_2 be bounded and boundedly invertible selfadjoint operators in a separable Hilbert space $\mathfrak{H}$. Let $\mathsf{i}(H_j) = \{\mathsf{i}_j^+, \mathsf{i}_j^-, \mathsf{i}_j^0, \mathsf{i}_j^\infty\}$ be the inertia of H_j, $j = 1, 2$, and assume that $\min\{\mathsf{i}_2^+, \mathsf{i}_1^-\} < \infty$ and that $H_1 \le H_2$. Then*

$$H_2^{-1} \le H_1^{-1} \quad \text{if and only if} \quad \mathsf{i}(H_1) = \mathsf{i}(H_2).$$

Very recently two extensions of Theorem 6 have been established in [15] for a general pair of selfadjoint operators and relations without any invertibility assumptions. For the present purposes we need the second main antitonicity theorem from [15], which reads as follows.

Theorem 7 *Let H_1 and H_2 be selfadjoint relations in a separable Hilbert space $\mathfrak{H}$ which are semibounded from below. Let $\mathsf{i}(H_j) = \{\mathsf{i}_j^+, \mathsf{i}_j^-, \mathsf{i}_j^0, \mathsf{i}_j^\infty\}$ be the inertia of H_j, $j = 1, 2$, and assume that $\mathsf{i}_1^- < \infty$ and that $H_1 \le H_2$. Then*

$$H_2^{-1} \le H_1^{-1} \quad \text{if and only if} \quad \mathsf{i}_1^- = \mathsf{i}_2^-.$$

The ordering appearing in Theorem 7 is defined via

$$H_1 \leq H_2 \quad \Leftrightarrow \quad 0 \leq (H_2 - aI)^{-1} \leq (H_1 - aI)^{-1},$$

where $a < \min\{\mu(H_1), \mu(H_2)\}$ is fixed and $\mu(H_i) \in \mathbb{R}$ stands for the lower bound of H_i, $i = 1, 2$. Notice that the conditions $H_1 \leq H_2$ and $\mathsf{i}_1^- < \infty$ imply $\mathsf{i}_2^- < \infty$; in particular these conditions already imply that the inverses H_1^{-1} and H_2^{-1} are also semibounded from below. For further facts on ordering of semibounded selfadjoint operators and relations the reader is referred to [15,42].

6.2 Cayley transforms

Define the linear fractional transformation $\mathcal{C}$, taking a linear relation A into a linear relation $\mathcal{C}(A)$, by

$$\mathcal{C}(A) = \{\{f + f', f - f'\} : \widehat{f} = \{f, f'\} \in A\} = -I + 2(I + A)^{-1}. \tag{35}$$

Clearly, $\mathcal{C}$ maps the (closed) linear relations one-to-one onto themselves, $\mathcal{C}^2 = I$, and

$$\mathcal{C}(A)^{-1} = \mathcal{C}(-A), \tag{36}$$

for every linear relation A. Moreover,

$$\begin{aligned} \operatorname{dom} \mathcal{C}(A) &= \operatorname{ran}(I + A), \quad \operatorname{ran} \mathcal{C}(A) = \operatorname{ran}(I - A), \\ \ker(\mathcal{C}(A) - I) &= \ker A, \quad \ker(\mathcal{C}(A) + I) = \operatorname{mul} A. \end{aligned}$$

In addition, $\mathcal{C}$ preserves closures, adjoints, componentwise sums, orthogonal sums, intersections, and inclusions. The relation $\mathcal{C}(A)$ is symmetric if and only if A is symmetric. It follows from (35) and

$$\|f + f'\|^2 - \|f - f'\|^2 = 4\operatorname{Re}(f', f) \tag{37}$$

that $\mathcal{C}$ gives a one-to-one correspondence between nonnegative (selfadjoint) linear relations and symmetric (respectively, selfadjoint) contractions. Observe the following mapping properties of $\mathcal{C}$ on the extended real line $\mathbb{R} \cup \{\pm\infty\}$:

$$\begin{aligned} &\mathcal{C}([0, 1]) = [0, 1]; \quad \mathcal{C}([-1, 0]) = [1, +\infty]; \\ &\mathcal{C}([1, +\infty]) = [-1, 0]; \quad \mathcal{C}([-\infty, -1]) = [-\infty, -1]. \end{aligned} \tag{38}$$

If H is a selfadjoint relation then

$$\mathsf{i}^-(I + H) = \mathsf{i}^-(\mathcal{C}(H) + I), \quad \mathsf{i}^-(I - H) = \mathsf{i}^-(\mathcal{C}(H)^{-1} + I),$$

and hence

$$\sigma(H) \cap (-\infty, -1) = \sigma(\mathfrak{C}(H)) \cap (-\infty, -1),$$
$$\sigma(H) \cap (1, +\infty) = \sigma(\mathfrak{C}(H)^{-1}) \cap (-\infty, -1) = \sigma(\mathfrak{C}(H)) \cap (-1, 0); \tag{39}$$

which can also be seen from (38).

6.3 M. G. Kreĭn's approach to the extension theory with a minimal negative index

In M. G. Kreĭn's approach to the extension theory of nonnegative operators the idea is to make a connection to the selfadjoint contractive extensions of a hermitian contraction T via the Cayley transform in (35). The extension of this approach to the present indefinite situation is based on the fact that the Cayley transform still reverses the ordering of selfadjoint extensions due to the antitonicity result formulated in Theorem 7 and the fact that in Theorem 5 $T \in \mathrm{Ext}_{T_1,\kappa}(-1,1)$ if and only if $T = T^* \supset T_1$ and $\nu_-(I \pm T) = \kappa_\mp$.

A semibounded symmetric relation A is said to be quasi-nonnegative if the associated form $a(f,f) := (f', f), \{f, f'\} \in A$, has a finite number of negative squares, i.e. every A-negative subspace $\mathfrak{L} \subset \mathrm{dom}\, A$ is finite dimensional. If the maximal dimension of A-negative subspaces is finite and equal to $\kappa \in \mathbb{Z}_+$, then A is said to be κ-nonnegative; the more precise notations $\nu_-(a)$, $\nu_-(A)$ are used to indicate the maximal number of negative squares of the form a and the relation A, respectively; here $\nu_-(a) = \nu_-(A)$. A selfadjoint extension $\widetilde{A}$ of A is said to be a κ-nonnegative extension of A if $\nu_-(\widetilde{A}) = \kappa$. The set of all such extension will be denoted by $\mathrm{Ext}_{A,\kappa}(0,\infty)$.

If A is a closed symmetric relation in the Hilbert space $\mathfrak{H}$ with $\kappa_-(A) < \infty$, then the subspace $\mathfrak{H}_1 := \mathrm{ran}\,(I + A)$ is closed, since the Cayley transform $T_1 = \mathfrak{C}(A)$ is a closed bounded symmetric operator in $\mathfrak{H}$ with $\mathrm{dom}\, T_1 = \mathfrak{H}_1$. Let P_1 be the orthogonal projection onto $\mathfrak{H}_1$ and let $P_2 = I - P_1$. Then the form

$$a_1(f,f) := (P_1 f', f), \quad \{f, f'\} \in A, \tag{40}$$

is symmetric and it has a finite number of negative squares.

Lemma 6 *Let A be a closed symmetric relation in $\mathfrak{H}$ with $\kappa_-(A) < \infty$ and let $T_1 = \mathfrak{C}(A)$. Then the form a_1 is given by*

$$a_1(f,f) = a(f,f) + \|P_2 f\|^2 \tag{41}$$

with $\nu_-(a_1) \le \nu_-(A)$. Moreover,

$$4a_1(f,f) = \|g\|^2 - \|T_{11} g\|^2, \quad 4a(f,f) = \|g\|^2 - \|T_1 g\|^2,$$

where $\{f, f'\} \in A$, $g = f + f'$, and $T_{11} = P_1 T_1$. In addition, $T_{21} = P_2 T_1$ satisfies $\|T_{21} g\|^2 = \|P_2 f\| = -(P_2 f, f')$.

Proof The formula (37) shows that if $T_1 = \mathcal{C}(A)$ and $\{f, f'\} \in A$, then

$$\|g\|^2 - \|T_1 g\|^2 = 4(f', f) = 4a(f, f), \quad g = f + f' \in \operatorname{dom} T_1 = \mathfrak{H}_1.$$

Moreover, $T_{21} g = P_2(f - f') = 2P_2 f = -2P_2 f'$ gives $(P_2 f', f) = -\|P_2 f\|^2$ and

$$\|T_{21} g\|^2 = -4(P_2 f', P_2 f) = -4(P_2 f', f).$$

In particular, (41) follows from

$$a(f, f) = (P_1 f', f) + (P_2 f', f) = a_1(f, f) - \|P_2 f\|^2.$$

Finally, (41) combined with $\|T_{21} g\|^2 = 4\|P_2 f\|^2$ leads to

$$4a_1(f, f) = \|g\|^2 - \|T_1 g\|^2 + \|T_{21} g\|^2 = \|g\|^2 - \|T_{11} g\|^2.$$

The main result in this section concerns the existence and a description of all selfadjoint extensions $\widetilde{A}$ of a symmetric relation A for which $\nu_-(\widetilde{A}) < \infty$ attains the minimal value $\nu_-(a_1)$. A criterion for the existence of such a selfadjoint extension is established, in which case all such extensions are described in a manner that is familiar from the case of nonnegative operators. To formulate the result assume that the selfadjoint quasi-contractive extensions T_m and T_M of T_1 as in Theorem 5 exist, and denote the corresponding selfadjoint relations A_F and A_K by

$$A_F = X(T_m) = -I + 2(I + T_m)^{-1}, \quad A_K = X(T_M) = -I + 2(I + T_M)^{-1}. \quad (42)$$

Theorem 8 *Let A be a closed symmetric relation in $\mathfrak{H}$ with $\nu_-(A) < \infty$ and denote $\kappa = \nu_-(a_1)\ (\leq \nu_-(A))$, where a_1 is given by (40). Then* $\operatorname{Ext}_{A,\kappa}(0, \infty)$ *is nonempty if and only if $\nu_-(A) = \kappa$. In this case A_F and A_K are well defined and they belong to* $\operatorname{Ext}_{A,\kappa}(0, \infty)$. *Moreover, the formula*

$$\widetilde{A} = -I + 2(I + T)^{-1} \quad (43)$$

gives a bijective correspondence between the quasi-contractive selfadjoint extensions $T \in \operatorname{Ext}_{T_1,\kappa}(-1, 1)$ of T_1 and the selfadjoint extensions $\widetilde{A} = \widetilde{A}^* \in \operatorname{Ext}_{A,\kappa}(0, \infty)$ of A. Furthermore, $\widetilde{A} = \widetilde{A}^* \in \operatorname{Ext}_{A,\kappa}(0, \infty)$ precisely when

$$A_K \leq \widetilde{A} \leq A_F, \quad (44)$$

or equivalently, when $A_F^{-1} \leq \widetilde{A}^{-1} \leq A_K^{-1}$, or

$$(A_F + I)^{-1} \leq (\widetilde{A} + I)^{-1} \leq (A_K + I)^{-1}. \quad (45)$$

The set $\mathrm{Ext}_{A^{-1},\kappa}(0,\infty)$ *is also nonempty and* $\widetilde{A} \in \mathrm{Ext}_{A,\kappa}(0,\infty)$ *if and only if* $\widetilde{A}^{-1} \in \mathrm{Ext}_{A^{-1},\kappa}(0,\infty)$. *The extreme selfadjoint extensions* A_F *and* A_K *of* A *are connected to those of* A^{-1} *via*

$$(A^{-1})_F = (A_K)^{-1}, \quad (A^{-1})_K = (A_F)^{-1}. \tag{46}$$

Proof Since $\nu_-(A) < \infty$, the Cayley transform $T_1 = \mathfrak{C}(A)$ defines a bounded symmetric operator in $\mathfrak{H}$ with $\mathfrak{H}_1 = \mathrm{dom}\, T_1 = \mathrm{ran}\,(I + A)$. It follows from Lemma 6 that

$$\nu_-(A) = \nu_-(a) = \nu_-(I - T_1^* T_1), \quad \nu_-(a_1) = \nu_-(I - T_{11}^2),$$

and therefore the condition $\nu_-(A) = \kappa$ is equivalent to solvability criterion (28) in Theorem 5. Moreover, $\widetilde{A}$ is a selfadjoint extension of A if and only if $T = \mathfrak{C}(\widetilde{A})$ is selfadjoint extension of T_1 and by Lemma 6 the equality $\nu_-(\widetilde{A}) = \nu_-(I - T^2)$ holds. Therefore, it follows from Theorem 5 that the set $\mathrm{Ext}_{A,\kappa}(0,\infty)$ is nonempty if and only if $\nu_-(A) = \kappa$ and in this case the formula (43) establishes a one-to-one correspondence between the sets $\mathrm{Ext}_{A,\kappa}(0,\infty)$ and $\mathrm{Ext}_{T_1,\kappa}(-1,1)$.

Next the characterizations (44) and (45) for the set $\mathrm{Ext}_{A,\kappa}(0,\infty)$ are established. Let $\widetilde{A} \in \mathrm{Ext}_{A,\kappa}(0,\infty)$ and let $T = \mathfrak{C}(\widetilde{A})$. According to Theorem 7 $T = \mathfrak{C}(\widetilde{A}) \in \mathrm{Ext}_{T_1,\kappa}(-1,1)$ if and only if T satisfies the inequalities $T_m \le T \le T_M$. It is clear from the formulas (42) and (43) that the inequalities $I + T_m \le I + T \le I + T_M$ are equivalent to the inequalities (45).

On the other hand, $\nu_-(I - T_{11}^2) = \nu_-(I - T^2)$ and hence the indices $\kappa_+ = \nu_-(I - T_{11}) = \nu_-(I - T)$ and $\kappa_- = \nu_-(I + T_{11}) = \nu_-(I + T)$ do not depend on $T = \mathfrak{C}(\widetilde{A})$; cf. (25). The mapping properties (39) of the Cayley transform imply that the number of eigenvalues of $\widetilde{A}$ on the open intervals $(-\infty, -1)$ and $(-1, 0)$ are also constant and equal to κ_- and κ_+, respectively. In particular, since $\kappa_- = \nu_-(I + T)$ is constant we can apply Theorem 6 to conclude that the inequalities $I + T_m \le I + T \le I + T_M$ are equivalent to

$$(I + T_M)^{-1} \le (I + T)^{-1} \le (I + T_m)^{-1},$$

which due to the formulas (42) and (43) can be rewritten as $A_F + I \le \widetilde{A} + I \le A_K + I$, or as $A_F \le \widetilde{A} \le A_K$. This proves (44). Since $\nu_-(\widetilde{A}) = \kappa = \kappa_- + \kappa_+$ is also constant, an application of Theorem 7 shows that the inequalities (44) are also equivalent to $A_F^{-1} \le \widetilde{A}^{-1} \le A_K^{-1}$.

As to the inverse A^{-1}, notice that $\nu_-(A^{-1}) = \nu_-(A)$. Moreover, since $A^{-1} = \mathfrak{C}(-T_1)$ it is clear that $\mathrm{ran}\,(I + A^{-1}) = \mathrm{dom}\, T_1$ and thus the form associated to A^{-1} via (40) satisfies $a_1^{(-1)}(f', f') = (P_1 f, f') = (P_1 f', f) = a_1(f, f)$. In particular, $\nu_-(a_1^{(-1)}) = \nu_-(a_1)$. Moreover, it is clear that $\nu_-(A^{-1}) = \nu_-(A)$. Consequently, the equality $\nu_-(A) = \nu_-(a_1)$ is equivalent to the equality $\nu_-(A^{(-1)}) = \nu_-(a_1^{(-1)})$. Furthermore, it is clear that $\widetilde{A} \in \mathrm{Ext}_{A,\kappa}(0,\infty)$ if and only if $\widetilde{A}^{-1} \in \mathrm{Ext}_{A^{-1},\kappa}(0,\infty)$.

Finally, the relations (46) are obtained from (31), (36), and (42).

It follows from Theorem 8 that the extensions $\widetilde{A} \in \mathrm{Ext}_{A,\kappa}(0,\infty)$ admit a uniform lower bound $\mu \le \mu(\widetilde{A})$ ($\mu \le 0$). This leads to the following inequalities for the resolvents.

Corollary 4 *With the assumptions as in Theorem 8 let $\nu_-(a_1) = \nu_-(A) < \infty$ and $\mu \le 0$ be a uniform lower bound for the extensions $\widetilde{A} \in \mathrm{Ext}_{A,\kappa}(0,\infty)$. Then the resolvents of these extensions satisfy the inequalities*

$$(A_F + a)^{-1} \le (\widetilde{A} + a)^{-1} \le (A_K + a)^{-1}, \quad a > -\mu. \tag{47}$$

Proof Let $T = \mathcal{C}(\widetilde{A}) \in \mathrm{Ext}_{T_1,\kappa}(-1,1)$ be the Cayley transform of the extension $\widetilde{A} \in \mathrm{Ext}_{A,\kappa}(0,\infty)$ and rewrite the resolvent of $\widetilde{A}$ in the form

$$(\widetilde{A} + a)^{-1} = \frac{1}{a-1} I - \frac{2}{(a-1)^2} \left(T + \frac{a+1}{a-1} \right)^{-1}.$$

Since $-a < \mu \le \mu(\widetilde{A})$, T admits precisely κ_- eigenvalues below the number $-(a+1)/(a-1) < -1$. Therefore the inequalities $T_m \le T \le T_M$ in Theorem 5 or, equivalently, the inequalities

$$T_m + \frac{a+1}{a-1} \le T + \frac{a+1}{a-1} \le T_M + \frac{a+1}{a-1}$$

imply the inequalities (47) by Theorem 6.

The antitonicity Theorems 6, 7 can be also used as follows. If the inequalities (44) and $A_F^{-1} \le \widetilde{A}^{-1} \le A_K^{-1}$ hold, then $\kappa = \nu_-(\widetilde{A}) = \nu_-(A_K) = \nu_-(A_F)$ is constant. If, in addition, (45) is satisfied, then it follows from (44) that $\kappa_- = \nu_-(I + \widetilde{A}) = \nu_-(I + A_K) = \nu_-(I + A_F)$ is constant, so that also $\kappa_+ = \nu_-(I - \widetilde{A}) = \nu_-(I - A_K) = \nu_-(I - A_F)$ is constant. However, in this case the equality $\nu_-(a_1) = \nu_-(A)$ need not hold and there can also be selfadjoint extensions $\widetilde{A}$ of A with

$$\nu_-(\widetilde{A}) = \nu_-(A_K) = \nu_-(A_F) > \nu_-(A) \ge \nu_-(a_1),$$

which neither satisfy the inequalities (44) and (45), nor the equalities $\nu_-(I + \widetilde{A}) = \kappa_-$ and $\nu_-(I - \widetilde{A}) = \kappa_+$. It is emphasized that the result in Theorem 8 characterizes all selfadjoint extensions in $\mathrm{Ext}_{A,\kappa}(0,\infty)$ under the minimal index condition $\kappa = \nu_-(a_1) = \nu_-(A)$.

In the case that A is nonnegative one has automatically $\kappa = \nu_-(a_1) = \nu_-(A) = 0$. Therefore, Theorem 8 is a precise generalization of the well-known characterization of the class $\mathrm{Ext}_A(0,\infty)$ (with $\kappa = 0$) due to M. G. Kreĭn [47] to the case of a finite negative (minimal) index $\kappa > 0$. The selfadjoint extensions A_F and A_K of A are called the Friedrichs (hard) and the Kreĭn–von Neumann (soft) extension, respectively; these notions go back to [36,56]. The extremal properties (47) of the Friedrichs and Kreĭn–von Neumann extensions were discovered by Kreĭn [47] in the case when A is a densely defined nonnegative operator. The case when $A \ge 0$ is not densely defined was considered by Ando and Nishio [5], and Coddington and de Snoo [22]. In the nonnegative case the formulas (46) can be found in [5,22]. Notice that in view of (43) and (44) the minimal solution of the completion problem for a nonnegative block operator A^0 can be also interpreted by means of the Kreĭn–von Neumann extension of the first column col (A_{11}, A_{21}) in (2); cf. [7, Section 4], [39, Section 4.9].

6.4 Kreĭn's uniqueness criterion

To establish a generalization of Kreĭn's uniqueness criterion for the equality $A_F = A_K$ in Theorem 8, i.e., for $\mathrm{Ext}_{A,\kappa}(0,\infty)$ to consists only of one extension, we first derive some general facts on J-contractions by means of their commutation properties.

Let $\mathfrak{H}_1$ and $\mathfrak{H}_2$ be Hilbert spaces with symmetries J_1 and J_2, respectively, and let $T \in [\mathfrak{H}_1, \mathfrak{H}_2]$ be a J-contraction, i.e., $J_1 - T^* J_2 T \geq 0$. Let D_T and D_{T^*} be the corresponding defect operators and let J_T and J_{T^*} be their signature operators as defined in Sect. 4. The first lemma connects the kernels of the defect operators D_T and D_{T^*}.

Lemma 7 *Let $T \in [\mathfrak{H}_1, \mathfrak{H}_2]$, let J_i be a symmetry in $\mathfrak{H}_i$, $i = 1, 2$, and let D_T and D_{T^*} be the defect operators of T and T^*, respectively. Then*

$$J_2 T(\ker D_T) = \ker D_{T^*}, \quad T^* J_2(\ker D_{T^*}) = \ker D_T. \tag{48}$$

In particular,

$$\ker D_T = \{0\} \text{ if and only if } \ker D_{T^*} = \{0\}.$$

Proof It suffices to show the first identity in (48). If $\varphi \in \ker D_T = \ker J_T D_T^2$, then the second identity in (19) implies that $J_2 T \varphi \in \ker J_{T^*} D_{T^*}^2 = \ker D_{T^*}$. Hence, $J_2 T(\ker D_T) \subset \ker D_{T^*}$. Conversely, let $\varphi \in \ker D_{T^*}$. Then $0 = J_{T^*} D_{T^*}^2 \varphi$ or, equivalently, $\varphi = J_2 T J_1 T^* \varphi$, and here $J_1 T^* \varphi \in \ker D_T$ by the first identity in (19). This proves the reverse inclusion.

Lemma 8 *Let the notations be as in Lemma 7. Then*

$$\mathrm{ran}\, T \cap \mathrm{ran}\, D_{T^*} = \mathrm{ran}\, T J_1 D_T = \mathrm{ran}\, D_{T^*} L_T,$$

where L_T is the link operator defined in Corollary 2.

Proof By the commutation formulas in Corollary 2 we have $\mathrm{ran}\, T J_1 D_T = \mathrm{ran}\, D_{T^*} L_T \subset \mathrm{ran}\, T \cap \mathrm{ran}\, D_{T^*}$. Hence, it suffices to prove the inclusion

$$\mathrm{ran}\, T \cap \mathrm{ran}\, D_{T^*} \subset \mathrm{ran}\, T J_1 D_T.$$

Suppose that $\varphi \in \mathrm{ran}\, T \cap \mathrm{ran}\, D_{T^*}$. Then Corollary 2 shows that $T^* J_2 \varphi = D_T f$ for some $f \in \mathfrak{D}_T$, while the second identity in (19) implies that

$$(J_2 - T J_1 T^*) J_2 \varphi = T J_1 D_T g,$$

for some $g \in \mathfrak{D}_T$. Therefore,

$$\varphi = (J_2 - TJ_1T^*)J_2\varphi + TJ_1T^*J_2\varphi = TJ_1D_Tg + TJ_1D_Tf = TJ_1D_T(g+f)$$

and this completes the proof.

We can now characterize J-isometric operators $T \in [\mathfrak{H}_1, \mathfrak{H}_2]$ as follows.

Proposition 3 *With the notations as in Lemma 7 the following statements are equivalent:*

(i) *T is J-isometric, i.e., $T^*J_2T = J_1$;*
(ii) $\ker T = \{0\}$ *and* $\operatorname{ran} T \cap \operatorname{ran} D_{T^*} = \{0\}$*;*
(iii) *for some, and equivalently for every, subspace $\mathfrak{L}$ with* $\operatorname{ran} J_2T \subset \overline{\mathfrak{L}}$ *one has*

$$\sup_{f\in\mathfrak{L}} \frac{|(f, T\varphi)|}{\|D_{T^*}f\|} = \infty \quad \textit{for every } \varphi \in \mathfrak{H}_1\backslash\{0\}, \tag{49}$$

i.e., there is no constant $0 \le C < \infty$ satisfying $|(f, T\varphi)| \le C\|D_{T^*}f\|$ for every $f \in \mathfrak{L}$, if $\varphi \neq 0$.

Proof (i) $\Rightarrow$ (iii) Let $\mathfrak{L}$ be an arbitrary subspace with $\operatorname{ran} J_2T \subset \overline{\mathfrak{L}}$. Assume that the supremum in (49) is finite for some $\varphi = J_1\psi \in \mathfrak{H}_1$. Then there exists $0 \le C < \infty$, such that

$$|(f, TJ_1\psi)| \le C\|D_{T^*}f\| \quad \text{for every } f \in \mathfrak{L}.$$

Since $\operatorname{ran} J_2T \subset \overline{\mathfrak{L}}$ and T is J-isometric, also the following inequality holds:

$$\|\psi\|^2 = (J_1T^*J_2T\psi, \psi) \le C\|D_{T^*}J_2T\psi\|. \tag{50}$$

By taking adjoints (and zero extension for L_{T^*}) in the second identity in Corollary 2 it is seen that $D_{T^*}J_2T\psi = L_{T^*}^*D_T\psi = 0$, since T is J-isometric. Hence (50) implies $\varphi = J_1\psi = 0$. Therefore (49) holds for every $\varphi \neq 0$.

(iii) $\Rightarrow$ (ii) Assume that (49) is satisfied with some subspace $\mathfrak{L}$. If (ii) does not hold, then either $\ker T \neq \{0\}$, in which case (49) does not hold for $0 \neq \varphi \in \ker T$, or $\operatorname{ran} T \cap \operatorname{ran} D_{T^*} \neq \{0\}$. However, then with $0 \neq T\varphi = D_{T^*}h$ the supremum in (49) is finite even if f varies over the whole space $\mathfrak{H}_2$. Thus, if (ii) does not hold then (49) fails to be true.

(ii) $\Rightarrow$ (i) Let $\operatorname{ran} T \cap \operatorname{ran} D_{T^*} = \{0\}$. Then by Lemma 8 one has $TJ_1D_T = 0$ and it follows from $\ker T = \{0\}$ that $D_T = 0$, i.e., T is isometric. This completes the proof.

After these preparations we are ready to prove the analog of Kreĭn's uniqueness criterion for the equality $T_m = T_M$ in the case of quasi-contractions appearing in Theorem 5.

Theorem 9 *Let the Hilbert space $\mathfrak{H}$ be decomposed as $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ and let $T_1 \in [\mathfrak{H}_1, \mathfrak{H}]$ be a symmetric quasi-contraction satisfying the condition (28) in Theorem 5. Then $T_m = T_M$ if and only if*

$$\sup_{f \in \mathfrak{H}_1} \frac{|(T_1 f, \varphi)|^2}{(|I - T_1^* T_1| f, f)} = \infty \quad \text{for every } \varphi \in \mathfrak{H}_2 \setminus \{0\}. \tag{51}$$

Proof Let $J = \operatorname{sign}(I - T_{11}^2)$. According to Theorem 5 there is $V \in [\mathfrak{D}_{T_{11}}, \mathfrak{H}_2]$, such that $T_{21} = V D_{T_{11}}$; moreover, V^* is a J-contraction, i.e., $I - V J V^* \geq 0$. This implies that

$$(T_1 f, \varphi) = (T_{21} f, \varphi) = (D_{T_{11}} f, V^* \varphi), \tag{52}$$

and a direct calculation shows that

$$I - T_1^* T_1 = I - T_{11}^2 - T_{21}^* T_{21} = J D_{T_{11}}^2 - D_{T_{11}} V^* V D_{T_{11}} = D_{T_{11}} D_V J_V D_V D_{T_{11}}. \tag{53}$$

By construction $D_V \in [\mathfrak{D}_{T_{11}}]$ and therefore $\operatorname{ran} D_V D_{T_{11}}$ is dense in $\mathfrak{D}_V = \overline{\operatorname{ran}}\, D_V$. Furthermore, since V^* is J-contractive it follows from Lemma 1 that $\nu_-(J_V) = \nu_-(J) = \nu_-(I - T_{11}^2)$ and, therefore, the assumption (28) shows that $\nu_-(J_V) = \nu_-(I - T_1^* T_1)$. Now according to Proposition 1 (ii) if follows from (53) that there is a unique J-unitary operator $C \in [\mathfrak{D}_{T_1}, \mathfrak{D}_V]$ such that $D_V D_{T_{11}} = C D_{T_1}$.

In view of (33) $T_m = T_M$ if and only if V^* is J-isometric. Since $\operatorname{ran} J V^* \subset \overline{\operatorname{ran}}\, D_{T_{11}}$, it follows from (i) and (iii) in Proposition 3 that $T := V^*$ satisfies the condition (49) with $\mathfrak{L} = \operatorname{ran} D_{T_{11}}$.

On the other hand, it follows from (53) and J-unitarity of $C \in [\mathfrak{D}_{T_1}, \mathfrak{D}_V]$ that

$$\|D_V D_{T_{11}}\| \leq \|C\|\, \|D_{T_1}\|, \quad \|D_{T_1}\| \leq \|C^{-1}\|\, \|D_V D_{T_{11}}\|.$$

By combining this equivalence between the norms of $\|D_{T_1}\|$ and $\|D_V D_{T_{11}}\|$ with the equality (52) one concludes that V^* satisfies the condition (49) precisely when T_1 satisfies the condition (51).

Remark 3 In the case of a hermitian contraction acting in a Hilbert space the criterion in Theorem 9 was proved by Kreĭn [47].

As to the geometric interpretation of the condition in Theorem 9, observe that if the supremum (51) is finite for some φ, then $T_{21}^* \varphi \in \operatorname{ran} D_{T_1}$ (see e.g. [38, Corollary 2]) and as the proof shows $D_{T_1} = D_{T_{11}} D_V C^{-*}$, which gives the equation $D_{T_{11}} V^* \varphi = D_{T_{11}} D_V C^{-*} v$ for some v. Consequently, $V^* \varphi = D_V C^{-*} v$ and hence again an application of Proposition 3 to V^*, now using items (i) and (ii), shows that (51) is equivalent to V^* being J-isometric. Here (see (33))

$$T_M - T_m = \begin{pmatrix} 0 & 0 \\ 0 & 2(I - V J V^*) \end{pmatrix}.$$

Recall that the minimal and maximal extension T_m and T_M of T_1 are determined via the minimal solutions $A_+ = I + T_m = S_-^* J_- S_-$ and $A_- = I - T_M = S_+^* J_+ S_+$ to the completion problems (26), where

$$S_- = |I + T_{11}|^{[-1/2]} T_{21}^*, \quad S_+ = |I - T_{11}|^{[-1/2]} T_{21}^*.$$

Here $Q_m := S_-^* J_- S_- = V(I - T_{11})JV^*$ and $Q_M := S_+^* J_+ S_+ = V(I + T_{11})JV^*$ appear when calculating the generalized Schur complements of the block operators A_+ and A_- using proper range inclusions; see Proposition 2 and (17). These two operators can be expressed either by limit values or by integrals as follows:

$$Q_m = T_{21}(I + T_{11})^{(-1)} T_{21}^* := \lim_{\varepsilon \uparrow 1} T_{21}(I + \varepsilon T_{11})^{-1} T_{21}^* = \int_{-\|T_{11}\|}^{\|T_{11}\|} \frac{T_{21}\, dE_t T_{21}^*}{1 + t},$$

$$Q_M = T_{21}(I - T_{11})^{(-1)} T_{21}^* := \lim_{\varepsilon \uparrow 1} T_{21}(I - \varepsilon T_{11})^{-1} T_{21}^* = \int_{-\|T_{11}\|}^{\|T_{11}\|} \frac{T_{21}\, dE_t T_{21}^*}{1 - t},$$

where ε is sufficiently close to 1 (to guarantee proper invertibility of indicated inverses) and E_t stands for the spectral family of T_{11}. With these notations the equality $T_m = T_M$ can be also rewritten in the form $Q_m - I = I - Q_M$, i.e. $2I = Q_m + Q_M = 2VJV^*$ or, equivalently,

$$\int_{-\|T_{11}\|}^{\|T_{11}\|} \frac{T_{21}\, dE_t T_{21}^*}{1 - t^2} = I. \tag{54}$$

In the special case of finite defect numbers ($\dim\,(\mathrm{dom}\, T_1)^{\perp} < \infty$) the condition (54) appears in Langer and Textorius [53, Theorem2.8]. Notice, that using the factorization $T_{21} = VD_{T_{11}}$ and the formula $I - T_{11}^2 = JD_{T_{11}^2}$ the condition (54) can immediately be rewritten in the form $VJV^* = I$.

The criterion in Theorem 9 can be translated to the situation of Theorem 8 via Cayley transform to get the analog of Kreĭn's uniqueness criterion for the equality $A_F = A_K$.

Corollary 5 *Let A be a closed symmetric relation in $\mathfrak{H}$ satisfying the condition $\nu_-(A) = \nu_-(a_1) < \infty$ in Theorem 8. Then the equality $A_F = A_K$ holds if and only if the following condition is fulfilled*:

$$\sup_{g \in \mathfrak{H}_1} \frac{|((A + I)^{-1} g, \varphi)|^2}{(|\widehat{A}| g, g)} = \infty \quad \text{for every } \varphi \in \ker\,(A^* + I)\backslash\{0\}, \tag{55}$$

where $\widehat{A} = (I + A)^{-*} A (I + A)^{-1}$ is a bounded selfadjoint operator in $\mathfrak{H}_1 = \mathrm{ran}\,(A + I)$.

Proof Let $T_1 = \mathfrak{C}(A)$ so that $\{f, f'\} \in A$ if and only if $\{f + f', 2f\} \in T_1 + I$; see (35). Then with $g = f + f' \in \mathrm{dom}\, T_1 = \mathfrak{H}_1$ and $\varphi \in \mathfrak{H}_2 = (\mathrm{dom}\, T_1)^{\perp}$ one has

$$(T_1 g, \varphi) = ((T_1 + I) g, \varphi) = 2((A + I)^{-1} g, \varphi).$$

Let $A_s = P_s A$ be the operator part of A; here P_s stands for the orthogonal projection onto mul $A = (\text{dom } A^*)^\perp = \ker(T_1 + I)$. Then the form $a(f, f) = (f', f)$ associated with A can be rewritten as $a(f, f) = (A_s f, f)$, $f \in \text{dom } A$, and thus

$$((I - T_1^* T_1)g, g) = 4(f', f) = 4(A_s(I + A)^{-1}g, (I + A)^{-1}g)),$$

where $2(I + A)^{-1} = T_1 + I$ is a bounded operator from $\mathfrak{H}_1$ into $\mathfrak{H}$. Then clearly $\widehat{A} = (I + A)^{-*} A_s (I + A)^{-1}$ is a bounded selfadjoint operator in $\mathfrak{H}_1$ and, moreover, $\nu_-(\widehat{A}) = \nu_-(a) = \nu_-(I - T_1^* T_1)$; see Lemma 6. Thus, it follows from Proposition 1 that there is a J-unitary operator C from $\overline{\text{ran}}\, \widehat{A}$ into $\mathfrak{D}_{T_1}$ such that $D_{T_1} = C|\widehat{A}|^{1/2}$. As in the proof of Theorem 8 this implies the equivalence of the conditions (51) and (55).

Observe that if A is nonnegative then with $\{f, f'\} \in A$ and $g = f + f' \in \mathfrak{H}_1$,

$$((A + I)^{-1}g, \varphi) = (f, \varphi), \quad (A_s(I + A)^{-1}g, (I + A)^{-1}g)) = (A_s f, f),$$

and, therefore, in this case the condition (55) can be rewritten as

$$\sup_{\{f, f'\} \in A} \frac{|(f, \varphi)|^2}{(f', f)} = \infty \quad \text{for every } \varphi \in \ker(A^* + I) \setminus \{0\},$$

the criterion which for a densely defined operator A was obtained in [47] and for a nonnegative relation A can be found in [38,39].

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References

1. Akhiezer, N.I., Glazman, I.M.: Theory of linear operators in Hilbert space. Translated from the third Russian edition by E. R. Dawson. Translation edited by W. N. Everitt. Monographs and Studies in Mathematics, 10, vol. II. Pitman (Advanced Publishing Program), Boston (1981)
2. Alonso, A., Simon, B.: The Birman–Kreĭn–Vishik theory of self-adjoint extensions of semibounded operators. J. Oper. Theory **4**, 251–270 (1980)
3. Anderson, W.N., Duffin, B.J.: Series and parallel addition of matrices. J. Math. Anal. Appl. **26**, 576–594 (1969)
4. Anderson, W.N., Trapp, G.E.: Shorted operators. II. SIAM J. Appl. Math. **28**, 60–71 (1975)
5. Ando, T., Nishio, K.: Positive self-adjoint extensions of positive symmetric operators. Tohóku Math. J. **22**, 65–75 (1970)
6. Antezana, J., Corach, G., Stojanoff, D.: Bilateral shorted operators and parallel sums. Linear Algebra Appl. **414**, 570–588 (2006)
7. Arlinskii, Yu.M.: Extremal extensions of sectorial linear relations. Mat. Stud. **7**, 81–96 (1997)
8. Arlinskiĭ, Yu.M.: Extremal extensions of a $C(\alpha)$-suboperator and their representations. Oper. Theory Adv. Appl. **162**, 47–69 (2006)
9. Arlinskiĭ, Yu., Belyi, S., Tsekanovskiĭ, E.: Conservative realizations of Herglotz–Nevanlinna functions. Operator Theory: Advances and Applications, vol. 217. Birkhäuser/Springer, Basel (2011)
10. Arlinskiĭ, Yu.M., Hassi, S., de Snoo, H.S.V.: Parametrization of contractive block-operator matrices and passive discrete-time systems. Complex Anal. Oper. Theory **1**, 211–233 (2007)

11. Arlinskiĭ Yu., Tsekanovskiĭ, E.: M. Kreĭn's research on semi-bounded operators, its contemporary developments, and applications. Modern analysis and applications. In: The Mark Krein Centenary Conference. Operator Theory and Related Topics, vol. 1, pp. 65–112. Oper. Theory Adv. Appl. **190**. Birkhäuser, Basel (2009)
12. Arsene, Gr., Gheondea, A.J.: Completing matrix contractions. Oper. Theory **7**, 179–189 (1982)
13. Arsene, Gr., Constantinescu, T., Gheondea, A.: Lifting of operators and prescribed numbers of negative squares. Michigan Math. J. **34**, 201–216 (1987)
14. Azizov, T.Ya., Iokhvidov, I.S.: Linear Operators in Spaces with Indefinite Metric. Wiley, New York (1989)
15. Behrndt, J., Hassi, S., de Snoo, H.S.V., Wietsma, H.L.: Antitonicity of the inverse for selfadjoint matrices, operators, and relations. Proc. Am. Math. Soc. **142**, 2783–2796 (2014)
16. Birman, M.S.: On the self-adjoint extensions of positive definite operators. Mat. Sb. **38**, 431–450 (1956)
17. Birman, M.S., Solomjak, M.Z.: Spectral theory of selfadjoint operators in Hilbert space. Translated from the 1980 Russian original by S. Khrushchev and V. Peller. In: Mathematics and its Applications (Soviet Series). D. Reidel Publishing Co., Dordrecht (1987)
18. Bognár, J.: Indefinite Inner Product Space. Springer, Berlin (1974)
19. Brasche, J.F., Neidhardt, H.: Some remarks on Kreĭn's extension theory. Math. Nachr. **165**, 159–181 (1994)
20. Bruk, V.M.: On one class of boundary value problems with a spectral parameter in the boundary condition (Russian) Mat. Sbornik **100**(2), 210–216 (1976)
21. Calkin, J.W.: Symmetric boundary conditions. Trans. AMS **45**(3), 369–442 (1939)
22. Coddington, E.A., de Snoo, H.S.V.: Positive selfadjoint extensions of positive symmetric subspaces. Math. Z. **159**, 203–214 (1978)
23. Constantinescu, T., Gheondea, A.: Minimal signature in lifting of operators. I. J. Oper. Theory **22**, 345–367 (1989)
24. Constantinescu, T., Gheondea, A.: Minimal signature in lifting of operators. II. J. Funct. Anal. **103**, 317–352 (1992)
25. Davis, Ch., Kahan, W.M., Weinberger, H.F.: Norm preserving dilations and their applications to optimal error bounds. SIAM J. Numer. Anal. **19**(3), 445–469 (1982)
26. Derkach, V.A., Hassi, S., Malamud, M.M., de Snoo, H.S.V.: Boundary relations and their Weyl families. Trans. Am. Math. Soc. **358**, 5351–5400 (2006)
27. Derkach, V.A., Hassi, S., Malamud, M.M., de Snoo, H.S.V.: Boundary triplets and Weyl functions. Recent developments. In: Operator Methods for Boundary Value Problems. Lecture Notes Series, vol. 404, pp. 161–220. London Mathematical Society, London (2012)
28. Derkach, V.A., Hassi, S., de Snoo, H.S.V.: Truncated moment problem in the class of generalized Nevanlinna functions. Math. Nachr. **285**(14–15), 1741–1769 (2012)
29. Derkach, V.A., Malamud, M.M.: Generalized resolvents and the boundary value problems for Hermitian operators with gaps. J. Funct. Anal. **95**(1), 1–95 (1991)
30. Derkach, V.A., Malamud, M.M.: On a generalization of the Kreĭn–Stieltjes class of functions (Russian). Izv. Akad. Nauk Armenii Mat. **26**(2), 115–137 (1991) **(Translation in J. Contemp. Math. Anal. 26(2), 16–37 (1991))**
31. Derkach, V.A., Malamud, M.M.: The extension theory of Hermitian operators and the moment problem. J. Math. Sci. **73**(2), 141–242 (1995)
32. Douglas, R.G.: On majorization, factorization and range inclusion of operators in Hilbert space. Proc. Am. Math. Soc. **17**, 413–416 (1966)
33. Dritschel, M.A.: A lifting theorem for bicontractions on Kreĭn spaces. J. Funct. Anal. **89**, 61–89 (1990)
34. Dritschel, M.A., Rovnyak, J.: Extension theorems for contraction operators on Kreĭn spaces. Extension and interpolation of linear operators and matrix functions. Oper. Theory Adv. Appl. **47**, 221–305 (1990)
35. Fillmore, P.A., Williams, J.P.: On operator ranges. Adv. Math. **7**, 254–281 (1971)
36. Friedrichs, K.O.: Spektraltheorie halbbeschränkter Operatoren und Anwendung auf die Spektralzerlegung von Differentialoperatoren. Math. Ann. **109**, 465–487 (1934)
37. Gorbachuk, V.I., Gorbachuk, M.L.: Boundary problems for differential operator equations (Russian). Naukova Dumka, Kiev (1984) **(Translated and revised from the 1984 Russian original: Mathematics and its Applications (Soviet Series), vol. 48. Kluwer Academic Publishers Group, Dordrecht (1991))**

38. Hassi, S.: On the Friedrichs and the Kreĭn–von Neumann extension of nonnegative relations. In: Contributions to management science, mathematics and modelling. Acta Wasaensia **122**, 37–54 (2004)
39. Hassi, S., Malamud, M.M., de Snoo, H.S.V.: On Kreĭn's extension theory of nonnegative operators. Math. Nachr. **274**(275), 40–73 (2004)
40. Hassi, S., Nordström, K.: Antitonicity of the inverse of selfadjoint operators. Schwerpunktprogramm der Deutschen Forschungsgemeinschaft. Forschungsbericht **364** (1992)
41. Hassi, S., Nordström, K.: Antitonicity of the inverse and J-contractivity. Oper. Theory Adv. Appl. **61**, 149–161 (1993)
42. Kato, T.: Perturbation Theory for Linear Operators. Springer, Berlin (1995)
43. Kochubeĭ, A.N.: Extensions of symmetric operators and symmetric binary relations (Russian). Math. Zametki **17**(1), 41–48 (1975) **(English translation in Math. Notes 17, 25–28 (1975))**
44. Kolmanovich, V.U., Malamud, M.M.: Extensions of sectorial operators and dual pair of contractions (Russian). Manuscript No 4428–85. Deposited at Vses. Nauchn-Issled, Inst. Nauchno-Techn. Informatsii, VINITI 19 04 85, Moscow. R ZH Mat **10B1144**, 1–57 (1985)
45. Kreĭn, M.G.: On Hermitian operators with defect indices (1, 1). Dokl. Akad. Nauk SSSR **43**, 339–342 (1944)
46. Kreĭn, M.G.: On resolvents of Hermitian operator with deficiency index (m, m). Dokl. Akad. Nauk SSSR **52**, 657–660 (1946)
47. Kreĭn, M.G.: The theory of self-adjoint extensions of semi-bounded Hermitian transformations and its applications. I. Mat. Sb. **20**(3), 431–495 (1947)
48. Kreĭn, M.G.: Completely continuous linear operators in function spaces with two norms. Akad. Nauk Ukrain. RSR. Zbirnik Prac' Inst. Mat. **9**, 104–129 (1947)
49. Kreĭn, M.G., Langer, H.: On defect subspaces and generalized resolvents of Hermitian operator in the space Π_κ (Russian). Funct. Anal. Appl. **5**(2), 59–71 (1971)
50. Kreĭn, M.G., Langer, H.: On defect subspaces and generalized resolvents of Hermitian operator in the space Π_κ (Russian). Funct. Anal. Appl. **5**(3), 54–69 (1971)
51. Kuzhel, A.V., Kuzhel, S.A.: Regular extensions of Hermitian operators **(Translated from the Russian by P. Malyshev and D. Malyshev. VSP, Utrecht (1998))**
52. Langer, H., Textorius, B.: On generalized resolvents and Q-functions of symmetric linear relations (subspaces) in Hilbert space. Pac. J. Math. **72**(1), 135–165 (1977)
53. Langer, H., Textorius, B.: Extensions of a bounded Hermitian operator T. Research Report LiTH-MAT-R-87-17, Department of Mathematics, Linköping University (1977)
54. Malamud, M.M.: Certain classes of extensions of a lacunary Hermitian operator. Ukrainian Math. J. **44**(2), 190–204 (1992)
55. Malamud, M.M.: On some classes of extensions of sectorial operators and dual pair of contractions. Oper. Theory Adv. Appl. **124**, 401–448 (2001)
56. von Neumann, J.: Allgemeine Eigenwerttheorie Hermitescher Funktionaloperatoren. Math. Ann. **102**, 49–131 (1929)
57. Pekarev, E.L.: Shorts of operators and some extremal problems. Acta Sci. Math. (Szeged) **56**, 147–163 (1992)
58. Rodman, L.: A note on indefinite Douglas' lemma. Operator theory in inner product spaces. Oper. Theory Adv. Appl. **175**, 225–229 (2007)
59. Shmul'yan, Yu.L.: A Hellinger operator integral (Russian). Mat. Sb. (N.S.) **49**(91), 381–430 (1959)
60. Shmul'yan, Yu.L.: Division in the class of J-expansive operators (Russian). Mat. Sb. (N.S.) **74**(116), 516–525 (1967)
61. Shmul'yan, Yu.L.: A question regarding inequalities between Hermitian operators (Russian). Mat. Zametki **49**, 138–141 (1991) **(English translation: Math. Notes 49, 423–425 (1991))**
62. Shmul'yan, Yu.L., Yanovskaya, R.N.: Blocks of a contractive operator matrix. Izv. Vyssh. Uchebn. Zaved. Mat. **7**, 72–75 (1981)
63. Vishik M.I.: On general boundary problems for elliptic differential equations (Russian). Trans. Moscow Math. Soc. **1**, 186–246 (1952) **(English translation: Am. Math. Soc. Transl. 24, 107–172 (1963))**

Completion of operators in Kreĭn spaces

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Abstract. A generalization of the well-known results of M.G. Kreĭn about the description of selfadjoint contractive extension of a hermitian contraction is obtained. This generalization concerns the situation, where the selfadjoint operator A and extensions $\widetilde{A}$ belong to a Kreĭn space or a Pontryagin space and their defect operators are allowed to have a fixed number of negative eigenvalues. Also a result of Yu.L. Shmul'yan on completions of nonnegative block operators is generalized for block operators with a fixed number of negative eigenvalues in a Kreĭn space.

This paper is a natural continuation of S. Hassi's and author's recent paper [5].

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1. Introduction

In 1947 M.G. Kreĭn published one of his famous papers [17] on a description of a nonnegative selfadjoint extensions of a densely defined nonnegative operator A in a Hilbert space. Namely, all nonnegative selfadjoint extensions $\widetilde{A}$ of A can be characterized by the following two inequalities:

$$(A_F + a)^{-1} \le (\widetilde{A} + a)^{-1} \le (A_K + a)^{-1}, \quad a > 0,$$

where the Friedrichs (hard) extension A_F and the Kreĭn-von Neumann (soft) extension A_K of A. He proved these results by transforming the problems the study of contractive operators.

The first result of the present paper is a generalization of a result due to Shmul'yan [19] on completions of nonnegative block operators where the result was applied for introducing so-called Hellinger operator integrals. This result was extended in [5] for block operators in a Hilbert space by allowing a fixed number of negative eigenvalues. In Section 2 this result is further extended to block operators which act in a Kreĭn space.

In paper [5] we studied classes of "quasi-contractive" symmetric operators T_1 allowing a finite number of negative eigenvalues for the associated defect operator $I-T_1^*T_1$, i.e., $\nu_-(I-T_1^*T_1)<\infty$ as well as "quasi-nonnegative" operators A with $\nu_-(A)<\infty$ and the existence and description of all possible selfadjoint extensions T and $\widetilde{A}$ of them which preserve the given negative indices $\nu_-(I-T^2)=\nu_-(I-T_1^*T_1)$ and $\nu_-(\widetilde{A})=\nu_-(A)$, and proved precise analogs of the above mentioned results of M.G. Kreĭn under a minimality condition on the negative indices $\nu_-(I-T_1^*T_1)$ and $\nu_-(A)$, respectively. It was an unexpected fact that when there is a solution then the solution set still contains a minimal solution and a maximal solution which then describe the whole solution set via two operator inequalities, just as in the original paper of M.G. Kreĭn. In this paper analogous results are established for "quasi-contractive" operators acting in a Kreĭn space; see Theorems 4.2, 5.7.

In Section 4 a first Kreĭn space analog of completion problem is formulated and a description of its solutions is found. Namely, we consider classes of "quasi-contractive" symmetric operators T_1 in a Kreĭn space with $\nu_-(I-T_1^*T_1)<\infty$ and we describe all possible selfadjoint (in the Kreĭn space sense) extensions T of T_1 which preserve the given negative index $\nu_-(I-T^*T)=\nu_-(I-T_1^*T_1)$. This problem is close to the completion problem studied in [5] and has a similar description for its solutions. For further history behind this problem see also [1, 2, 3, 7, 8, 9, 10, 11, 12, 14, 15, 16, 20].

The main result of the present paper is proved in Section 5. Namely, we consider classes of "quasi-contractive" symmetric operators T_1 in a Kreĭn space $(\mathfrak{H},J)$ with

$$\nu_-[I-T_1^{[*]}T_1]:=\nu_-(J(I-T_1^{[*]}T_1))<\infty \tag{1.1}$$

and we establish a solvability criterion and a description of all possible selfadjoint extensions T of T_1 (in the Kreĭn space sense) which preserve the given negative index $\nu_-[I-T^{[*]}T]=\nu_-[I-T_1^{[*]}T_1]$. It should be pointed out that in this more general setting the descriptions involve so-called link operator L_T which was introduced by Arsene, Constantintscu and Gheondea in [3] (see also [2, 7, 8, 18]).

2. A completion problem for block operators in Kreĭn spaces

By definition the modulus $|C|$ of a closed operator C is the nonnegative selfadjoint operator $|C|=(C^*C)^{1/2}$. Every closed operator admits a polar decomposition $C=U|C|$, where U is a (unique) partial isometry with the initial space $\overline{\operatorname{ran}}\,|C|$ and the final space $\overline{\operatorname{ran}}\,C$, cf. [13]. For a selfadjoint operator $H=\int_{\mathbb{R}} t\,dE_t$ in a Hilbert space $\mathfrak{H}$ the partial isometry U can be identified with the signature operator, which can be taken to be unitary: $J=\operatorname{sign}(H)=\int_{\mathbb{R}}\operatorname{sign}(t)\,dE_t$, in which case one should define $\operatorname{sign}(t)=1$ if $t\geq 0$ and otherwise $\operatorname{sign}(t)=-1$.

Let $\mathcal{H}$ be a Hilbert space, and let $J_{\mathcal{H}}$ be a signature operator in it, i.e., $J_{\mathcal{H}} = J_{\mathcal{H}}^* = J_{\mathcal{H}}^{-1}$. We interpret the space $\mathcal{H}$ as a Kreĭn space $(\mathcal{H}, J_{\mathcal{H}})$ (see [4, 6]) in which the indefinite scalar product is defined by the equality

$$[\varphi, \psi]_{\mathcal{H}} = (J_{\mathcal{H}}\varphi, \psi)_{\mathcal{H}}.$$

Let us introduce a partial ordering for selfadjoint Kreĭn space operators. For selfadjoint operators A and B with the same domains $A \geq_J B$ if and only if $[(A - B)f, f] \geq 0$ for all $f \in \operatorname{dom} A$. If not otherwise indicated the word "smallest" means the smallest operator in the sense of this partial ordering.

Consider a bounded incomplete block operator

$$A^0 = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & * \end{pmatrix} \begin{pmatrix} (\mathfrak{H}_1, J_1) \\ (\mathfrak{H}_2, J_2) \end{pmatrix} \to \begin{pmatrix} (\mathfrak{H}_1, J_1) \\ (\mathfrak{H}_2, J_2) \end{pmatrix} \tag{2.1}$$

in the Kreĭn space $\mathfrak{H} = (\mathfrak{H}_1 \oplus \mathfrak{H}_2, J)$, where $(\mathfrak{H}_1, J_1)$ and $(\mathfrak{H}_2, J_2)$ are Kreĭn spaces with fundamental symmetries J_1 and J_2, and $J = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix}$.

Theorem 2.1. *Let $\mathfrak{H} = (\mathfrak{H}_1 \oplus \mathfrak{H}_2, J)$ be an orthogonal decomposition of the Kreĭn space $\mathfrak{H}$ and let A^0 be an incomplete block operator of the form* (2.1). *Assume that $A_{11} = A_{11}^{[*]}$ and $A_{21} = A_{12}^{[*]}$ are bounded, the numbers of negative squares of the quadratic form $[A_{11}f, f]$ $(f \in \operatorname{dom} A_{11})$ $\nu_-[A_{11}] := \nu_-(J_1 A_{11}) = \kappa < \infty$, where $\kappa \in \mathbb{Z}_+$, and let us introduce $J_{11} := \operatorname{sign}(J_1 A_{11})$ the (unitary) signature operator of $J_1 A_{11}$. Then:*

(i) *There exists a completion $A \in [(\mathfrak{H}, J)]$ of A^0 with some operator $A_{22} = A_{22}^{[*]} \in [(\mathfrak{H}_2, J_2)]$ such that $\nu_-[A] = \nu_-[A_{11}] = \kappa$ if and only if*

$$\operatorname{ran} J_1 A_{12} \subset \operatorname{ran} |A_{11}|^{1/2}.$$

(ii) *In this case the operator $S = |A_{11}|^{[-1/2]} J_1 A_{12}$, where $|A_{11}|^{[-1/2]}$ denotes the (generalized) Moore-Penrose inverse of $|A_{11}|^{1/2}$, is well defined and $S \in [(\mathfrak{H}_2, J_2), (\mathfrak{H}_1, J_1)]$. Moreover, $S^{[*]} J_1 J_{11} S$ is the "smallest" operator in the solution set*

$$\mathcal{A} := \left\{ A_{22} = A_{22}^{[*]} \in [(\mathfrak{H}_2, J_2)] : \ A = (A_{ij})_{i,j=1}^2 : \nu_-[A] = \kappa \right\}$$

and this solution set admits a description

$$\mathcal{A} = \Big\{ A_{22} \in [(\mathfrak{H}_2, J_2)] : \ A_{22} = J_2(S^* J_{11} S + Y) = S^{[*]} J_1 J_{11} S + J_2 Y, \quad \text{where } Y = Y^* \geq 0 \Big\}.$$

Proof. Let us introduce a block operator

$$\widetilde{A}^0 = \begin{pmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} \\ \widetilde{A}_{21} & * \end{pmatrix} = \begin{pmatrix} J_1 A_{11} & J_1 A_{12} \\ J_2 A_{21} & * \end{pmatrix}.$$

The blocks of this operator satisfy the identities $\widetilde{A}_{11} = \widetilde{A}_{11}^*$, $\widetilde{A}_{21}^* = \widetilde{A}_{12}$ and

$$\operatorname{ran} J_1 A_{11} = \operatorname{ran} \widetilde{A}_{11} \subset \operatorname{ran} |\widetilde{A}_{11}|^{1/2} = \operatorname{ran} (\widetilde{A}_{11}^* \widetilde{A}_{11})^{1/4} = \operatorname{ran} (A_{11}^* A_{11})^{1/4} = \operatorname{ran} |A_{11}|^{1/2}.$$

Then due to [5, Theorem 1] a description of all selfadjoint operator completions of $\widetilde{A}^0$ admits representation $\widetilde{A} = \begin{pmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} \\ \widetilde{A}_{21} & \widetilde{A}_{22} \end{pmatrix}$ with $\widetilde{A}_{22} = \widetilde{S}^* J_{11} \widetilde{S} + Y$, where $\widetilde{S} = |\widetilde{A}_{11}|^{[-1/2]} \widetilde{A}_{12}$ and $Y = Y^* \geq 0$.

This yields description for the solutions of the completion problem. The set of completions has the form $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, where

$$\begin{aligned} A_{22} = J_2 \widetilde{A}_{22} &= J_2 A_{21} J_1 |A_{11}|^{[-1/2]} J_{11} |A_{11}|^{[-1/2]} J_1 A_{12} + J_2 Y \\ &= J_2 S^* J_{11} S + J_2 Y = S^{[*]} J_1 J_{11} S + J_2 Y. \end{aligned}$$ □

3. Some inertia formulas

Some simple inertia formulas are now recalled. The factorization $H = B^{[*]} E B$ clearly implies that $\nu_\pm[H] \leq \nu_\pm[E]$, cf. (1.1). If H_1 and H_2 are selfadjoint operators in a Kreĭn space, then

$$H_1 + H_2 = \begin{pmatrix} I \\ I \end{pmatrix}^{[*]} \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} \begin{pmatrix} I \\ I \end{pmatrix}$$

shows that $\nu_\pm[H_1 + H_2] \leq \nu_\pm[H_1] + \nu_\pm[H_2]$. Consider the selfadjoint block operator $H \in [(\mathfrak{H}_1, J_1) \oplus (\mathfrak{H}_2, J_2)]$, where $J_i = J_i^* = J_i^{-1}$, $(i = 1, 2)$ of the form

$$H = H^{[*]} = \begin{pmatrix} A & B^{[*]} \\ B & I \end{pmatrix},$$

By applying the above mentioned inequalities shows that

$$\nu_\pm[A] \leq \nu_\pm[A - B^{[*]} B] + \nu_\pm(J_2). \tag{3.1}$$

Assuming that $\nu_-[A - B^* J_2 B]$ and $\nu_-(J_2)$ are finite, the question when $\nu_-[A]$ attains its maximum in (3.1), or equivalently, $\nu_-[A - B^* J_2 B] \geq \nu_-[A] - \nu_-(J_2)$ attains its minimum, turns out to be of particular interest. The next result characterizes this situation as an application of Theorem 2.1. Recall that if $J_1 A = J_A |A|$ is the polar decomposition of $J_1 A$, then one can interpret $\mathfrak{H}_A = (\overline{\operatorname{ran}}\, J_1 A, J_A)$ as a Kreĭn space generated on $\overline{\operatorname{ran}}\, J_1 A$ by the fundamental symmetry $J_A = \operatorname{sign}(J_1 A)$.

Theorem 3.1. *Let $A \in [(\mathfrak{H}_1, J_1)]$ be selfadjoint, $B \in [(\mathfrak{H}_1, J_1), (\mathfrak{H}_2, J_2)]$, $J_i = J_i^* = J_i^{-1} \in [\mathfrak{H}_i]$, $(i = 1, 2)$, and assume that $\nu_-[A], \nu_-(J_2) < \infty$. If the equality*

$$\nu_-[A] = \nu_-[A - B^{[*]} B] + \nu_-(J_2)$$

holds, then $\operatorname{ran} J_1 B^{[*]} \subset \operatorname{ran} |A|^{1/2}$ *and* $J_1 B^{[*]} = |A|^{1/2} K$ *for a unique operator* $K \in [(\mathfrak{H}_2, J_2), \mathfrak{H}_A]$ *which is J-contractive:* $J_2 - K^* J_A K \geq 0$.

Conversely, if $B^{[*]} = |A|^{1/2} K$ *for some J-contractive operator* $K \in [(\mathfrak{H}_2, J_2), \mathfrak{H}_A]$*, then the equality* (3.1) *is satisfied.*

Proof. Assume that (3.1) is satisfied. The factorization

$$H = \begin{pmatrix} A & B^{[*]} \\ B & I \end{pmatrix} = \begin{pmatrix} I & B^{[*]} \\ 0 & I \end{pmatrix} \begin{pmatrix} A - B^{[*]}B & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ B & I \end{pmatrix}$$

shows that $\nu_-[H] = \nu_-[A - B^{[*]}B] + \nu_-(J_2)$, which combined with the equality (3.1) gives $\nu_-[H] = \nu_-[A]$. Therefore, by Theorem 2.1 one has $\operatorname{ran} J_1 B^{[*]} \subset \operatorname{ran} |A|^{1/2}$ and this is equivalent to the existence of a unique operator $K \in [(\mathfrak{H}_2, J_2), \mathfrak{H}_A]$ such that $J_1 B^{[*]} = |A|^{1/2} K$; i.e. $K = |A|^{[-1/2]} J_1 B^{[*]}$. Furthermore, $K^{[*]} J_1 J_A K \leq_{J_2} I$ by the minimality property of $K^{[*]} J_1 J_A K$ in Theorem 2.1, in other words K is a J-contraction.

Converse, if $J_1 B^{[*]} = |A|^{1/2} K$ for some J-contractive operator $K \in [(\mathfrak{H}_2, J_2), \mathfrak{H}_A]$, then clearly $\operatorname{ran} J_1 B^{[*]} \subset \operatorname{ran} |A|^{1/2}$. By Theorem 2.1 the completion problem for H^0 has solutions with the minimal solution $S^{[*]} J_1 J_A S$, where

$$S = |A|^{[-1/2]} J_1 B^{[*]} = |A|^{[-1/2]} |A|^{1/2} K = K.$$

Furthermore, by J-contractivity of K one has $K^{[*]} J_1 J_A K \leq_{J_2} I$, i.e. I is also a solution and thus $\nu_-[H] = \nu_-[A]$ or, equivalently, the equality (3.1) is satisfied. □

4. A pair of completion problems in a Kreĭn space

In this section we introduce and describe the solutions of a Kreĭn space version of a completion problem that was treated in [5].

Let $(\mathfrak{H}_i, (J_i \cdot, \cdot))$ and $(\mathfrak{H}, (J \cdot, \cdot))$ be Kreĭn spaces, where $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$, $J = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix}$, and J_i are fundamental symmetries ($i = 1, 2$), let $T_{11} = T_{11}^{[*]} \in [(\mathfrak{H}_1, J_1)]$ be an operator such that $\nu_-(I - T_{11}^* T_{11}) = \kappa < \infty$. Denote $\widetilde{T}_{11} = J_1 T_{11}$, then $\widetilde{T}_{11} = \widetilde{T}_{11}^*$ in the Hilbert space $\mathfrak{H}_1$. Rewrite $\nu_-(I - T_{11}^* T_{11}) = \nu_-(I - \widetilde{T}_{11}^2)$. Denote

$$J_+ = \operatorname{sign}(I - \widetilde{T}_{11}), \ J_- = \operatorname{sign}(I + \widetilde{T}_{11}), \text{ and } J_{11} = \operatorname{sign}(I - \widetilde{T}_{11}^2),$$

and let $\kappa_+ = \nu_-(J_+)$ and $\kappa_- = \nu_-(J_-)$. It is easy to get that $J_{11} = J_- J_+ = J_+ J_-$. Moreover, there is an equality $\kappa = \kappa_- + \kappa_+$ (see [5, Lemma 5.1]). We recall the results for the operator $\widetilde{T}_{11}$ from the paper [5] and after that reformulate them for the operator T_{11}. We recall completion problem and its solutions that was investigated in a Hilbert space setting in [5]. The problem concerns the existence and a description of selfadjoint operators $\widetilde{T}$ such that $\widetilde{A}_+ = I + \widetilde{T}$ and $\widetilde{A}_- = I - \widetilde{T}$ solve the corresponding completion problems

$$\widetilde{A}_\pm^0 = \begin{pmatrix} I \pm \widetilde{T}_{11} & \pm \widetilde{T}_{21}^* \\ \pm \widetilde{T}_{21} & * \end{pmatrix}, \tag{4.1}$$

under *minimal index conditions* $\nu_-(I + \widetilde{T}) = \nu_-(I + \widetilde{T}_{11})$, $\nu_-(I - \widetilde{T}) = \nu_-(I - \widetilde{T}_{11})$, respectively. The solution set is denoted by $\operatorname{Ext}_{\widetilde{T}_1, \kappa}(-1, 1)$.

The next theorem gives a general solvability criterion for the completion problem (4.1) and describes all solutions to this problem.

Theorem 4.1. ([5, Theorem 5]) *Let* $\widetilde{T}_1 = \begin{pmatrix} \widetilde{T}_{11} \\ \widetilde{T}_{21} \end{pmatrix} : \mathfrak{H}_1 \to \begin{pmatrix} \mathfrak{H}_1 \\ \mathfrak{H}_2 \end{pmatrix}$ *be a symmetric operator with* $\widetilde{T}_{11} = \widetilde{T}_{11}^* \in [\mathfrak{H}_1]$ *and* $\nu_-(I - \widetilde{T}_{11}^2) = \kappa < \infty$, *and let* $J_{11} = \operatorname{sign}(I - \widetilde{T}_{11}^2)$. *Then the completion problem for* $\widetilde{A}_\pm^0$ *in* (4.1) *has a solution* $I \pm \widetilde{T}$ *for some* $\widetilde{T} = \widetilde{T}^*$ *with* $\nu_-(I - \widetilde{T}^2) = \kappa$ *if and only if the following condition is satisfied:*

$$\nu_-(I - \widetilde{T}_{11}^2) = \nu_-(I - \widetilde{T}_1^* \widetilde{T}_1). \tag{4.2}$$

If this condition is satisfied then the following facts hold:

(i) *The completion problems for* $\widetilde{A}_\pm^0$ *in* (4.1) *have minimal solutions* $\widetilde{A}_\pm$.

(ii) *The operators* $\widetilde{T}_m := \widetilde{A}_+ - I$ *and* $\widetilde{T}_M := I - \widetilde{A}_- \in \operatorname{Ext}_{\widetilde{T}_1,\kappa}(-1,1)$.

(iii) *The operators* $\widetilde{T}_m$ *and* $\widetilde{T}_M$ *have the block form*

$$\begin{aligned} \widetilde{T}_m &= \begin{pmatrix} \widetilde{T}_{11} & D_{\widetilde{T}_{11}} V^* \\ V D_{\widetilde{T}_{11}} & -I + V(I - \widetilde{T}_{11}) J_{11} V^* \end{pmatrix}, \\ \widetilde{T}_M &= \begin{pmatrix} \widetilde{T}_{11} & D_{\widetilde{T}_{11}} V^* \\ V D_{\widetilde{T}_{11}} & I - V(I + \widetilde{T}_{11}) J_{11} V^* \end{pmatrix}, \end{aligned} \tag{4.3}$$

where $D_{\widetilde{T}_{11}} := |I - \widetilde{T}_{11}^2|^{1/2}$ *and* V *is given by* $V := \operatorname{clos}(\widetilde{T}_{21} D_{\widetilde{T}_{11}}^{[-1]})$.

(iv) *The operators* $\widetilde{T}_m$ *and* $\widetilde{T}_M$ *are extremal extensions of* $\widetilde{T}_1$:

$$\widetilde{T} \in \operatorname{Ext}_{\widetilde{T}_1,\kappa}(-1,1) \quad \text{iff} \quad \widetilde{T} = \widetilde{T}^* \in [\mathfrak{H}], \quad \widetilde{T}_m \le \widetilde{T} \le \widetilde{T}_M.$$

(v) *The operators* $\widetilde{T}_m$ *and* $\widetilde{T}_M$ *are connected via*

$$(-\widetilde{T})_m = -\widetilde{T}_M, \quad (-\widetilde{T})_M = -\widetilde{T}_m.$$

For what follows it is convenient to reformulate the above theorem in a Kreĭn space setting. Consider the Kreĭn space $(\mathfrak{H}, J)$ and a selfadjoint operator T in this space. Now the problem concerns selfadjoint operators $A_+ = I + T$ and $A_- = I - T$ in the Kreĭn space $(\mathfrak{H}, J)$ that solve the completion problems

$$A_\pm^0 = \begin{pmatrix} I \pm T_{11} & \pm T_{21}^{[*]} \\ \pm T_{21} & * \end{pmatrix}, \tag{4.4}$$

under *minimal index conditions* $\nu_-(I+JT) = \nu_-(I+J_1T_{11})$ and $\nu_-(I-JT) = \nu_-(I - J_1T_{11})$, respectively. The set of solutions T to the problem (4.4) will be denoted by $\operatorname{Ext}_{J_2T_1,\kappa}(-1,1)$.

Denote

$$T_1 = \begin{pmatrix} T_{11} \\ T_{21} \end{pmatrix} : (\mathfrak{H}_1, J_1) \to \begin{pmatrix} (\mathfrak{H}_1, J_1) \\ (\mathfrak{H}_2, J_2) \end{pmatrix}, \tag{4.5}$$

so that T_1 is symmetric (nondensely defined) operator in the Kreĭn space $[(\mathfrak{H}_1, J_1)]$, i.e. $T_{11} = T_{11}^{[*]}$.

Theorem 4.2. *Let T_1 be a symmetric operator in a Kreĭn space sense as in (4.5) with $T_{11} = T_{11}^{[*]} \in [(\mathfrak{H}_1, J_1)]$ and $\nu_-(I - T_{11}^* T_{11}) = \kappa < \infty$, and let $J = \operatorname{sign}(I - T_{11}^* T_{11})$. Then the completion problems for $A_\pm^0$ in (4.4) have a solution $I \pm T$ for some $T = T^{[*]}$ with $\nu_-(I - T^*T) = \kappa$ if and only if the following condition is satisfied:*

$$\nu_-(I - T_{11}^* T_{11}) = \nu_-(I - T_1^* T_1). \tag{4.6}$$

If this condition is satisfied then the following facts hold:

(i) *The completion problems for $A_\pm^0$ in (4.4) have "minimal"(J_2-minimal) solutions $A_\pm$.*

(ii) *The operators $T_m := A_+ - J$ and $T_M := J - A_- \in \operatorname{Ext}_{J_2T_1,\kappa}(-1,1)$.*

(iii) *The operators T_m and T_M have the block form*

$$\begin{aligned} T_m &= \begin{pmatrix} T_{11} & J_1 D_{T_{11}} V^* \\ J_2 V D_{T_{11}} & -J_2 + J_2 V (I - J_1 T_{11}) J_{11} V^* \end{pmatrix}, \\ T_M &= \begin{pmatrix} T_{11} & J_1 D_{T_{11}} V^* \\ J_2 V D_{T_{11}} & J_2 - J_2 V (I + J_1 T_{11}) J_{11} V^* \end{pmatrix}, \end{aligned} \tag{4.7}$$

where $D_{T_{11}} := |I - T_{11}^* T_{11}|^{1/2}$ and V is given by $V := \operatorname{clos}(J_2 T_{21} D_{T_{11}}^{[-1]})$.

(iv) *The operators T_m and T_M are J_2-extremal extensions of T_1:*

$$T \in \operatorname{Ext}_{J_2T_1,\kappa}(-1,1) \quad \textit{iff} \quad T = T^{[*]} \in [(\mathfrak{H}, J)], \quad T_m \leq_{J_2} T \leq_{J_2} T_M.$$

(v) *The operators T_m and T_M are connected via*

$$(-T)_m = -T_M, \quad (-T)_M = -T_m.$$

Proof. The proof is obtained by systematic use of the equivalence that T is a selfadjoint operator in a Kreĭn space if and only if $\widetilde{T}$ is a selfadjoint in a Hilbert space. In particular, T gives solutions to the completion problems (4.4) if and only if $\widetilde{T}$ solves the completion problems (4.4). In view of

$$I - T_{11}^* T_{11} = I - T_{11}^* J J T_{11} = I - \widetilde{T}_{11}^2,$$

we are getting formula (4.6) from (4.2). Then formula (4.7) follows by multiplying the operators in (4.3) by the fundamental symmetry. □

5. Completion problem in a Pontryagin space

5.1. Defect operators and link operators

Let $(\mathfrak{H}, (\cdot,\cdot))$ be a Hilbert space and let J be a symmetry in $\mathfrak{H}$, i.e. $J = J^* = J^{-1}$, so that $(\mathfrak{H}, (J\cdot,\cdot))$, becomes a Pontryagin space. Then associate with $T \in [\mathfrak{H}]$ the corresponding defect and signature operators

$$D_T = |J - T^* J T|^{1/2}, \quad J_T = \operatorname{sign}(J - T^* J T), \quad \mathfrak{D}_T = \overline{\operatorname{ran}}\, D_T,$$

where the so-called defect subspace $\mathfrak{D}_T$ can be considered as a Pontryagin space with the fundamental symmetry J_T. Similar notations are used with T^*:

$$D_{T^*} = |J - T J T^*|^{1/2}, \quad J_{T^*} = \operatorname{sign}(J - T J T^*), \quad \mathfrak{D}_{T^*} = \overline{\operatorname{ran}}\, D_{T^*}.$$

By definition $J_T D_T^2 = J - T^* JT$ and $J_T D_T = D_T J_T$ with analogous identities for D_{T^*} and J_{T^*}. In addition,

$$(J - T^* JT)JT^* = T^* J(J - TJT^*),\ (J - TJT^*)JT = TJ(J - T^* JT).$$

Recall that $T \in [\mathfrak{H}]$ is said to be a J-contraction if $J - T^* JT \geq 0$, i.e. $\nu_-(J - T^* JT) = 0$. If, in addition, T^* is a J-contraction, T is termed as a J-bicontraction.

For the following consideration an indefinite version of the commutation relation of the form $TD_T = D_{T^*} T$ is needed; these involve so-called link operators introduced in [3, Section 4] (see also [5]).

Definition 5.1. There exist unique operators $L_T \in [\mathfrak{D}_T, \mathfrak{D}_{T^*}]$ and $L_{T^*} \in [\mathfrak{D}_{T^*}, \mathfrak{D}_T]$ such that

$$D_{T^*} L_T = TJD_T \upharpoonright \mathfrak{D}_T, \quad D_T L_{T^*} = T^* JD_{T^*} \upharpoonright \mathfrak{D}_{T^*}; \tag{5.1}$$

in fact, $L_T = D_{T^*}^{[-1]} TJD_T \upharpoonright \mathfrak{D}_T$ and $L_{T^*} = D_T^{[-1]} T^* JD_{T^*} \upharpoonright \mathfrak{D}_{T^*}$.

The following identities can be obtained with direct calculations; see [3, Section 4]:

$$\begin{aligned} L_T^* J_{T^*} \upharpoonright \mathfrak{D}_{T^*} &= J_T L_{T^*}; \\ (J_T - D_T J D_T) \upharpoonright \mathfrak{D}_T &= L_T^* J_{T^*} L_T; \\ (J_{T^*} - D_{T^*} J D_{T^*}) \upharpoonright \mathfrak{D}_{T^*} &= L_{T^*}^* J_T L_{T^*}. \end{aligned} \tag{5.2}$$

Now let T be selfadjoint in Pontryagin space $(\mathfrak{H}, J)$, i.e. $T^* = JTJ$. Then connections between D_{T^*} and D_T, J_{T^*} and J_T, L_{T^*} and L_T can be established.

Lemma 5.2. *Assume that* $T^* = JTJ$. *Then* $D_T = |I - T^2|^{1/2}$ *and the following equalities hold:*

$$D_{T^*} = JD_T J, \tag{5.3}$$

in particular,

$$\mathfrak{D}_{T^*} = J\mathfrak{D}_T \text{ and } \mathfrak{D}_T = J\mathfrak{D}_{T^*};$$

$$J_{T^*} = JJ_T J; \tag{5.4}$$

$$L_{T^*} = JL_T J. \tag{5.5}$$

Proof. The defect operator of T can be calculated by the formula

$$D_T = \left(\left(I - (T^*)^2\right) JJ(I - T^2)\right)^{1/4} = \left(\left(I - (T^*)^2\right)(I - T^2)\right)^{1/4}.$$

Then

$$\begin{aligned} D_{T^*} &= \left(J\left(I - (T^*)^2\right)(I - T^2)J\right)^{1/4} = J\left(\left(I - (T^*)^2\right)(I - T^2)\right)^{1/4} J \\ &= JD_T J \end{aligned}$$

i.e. (5.3) holds. This implies

$$J\mathfrak{D}_{T^*} \subset \mathfrak{D}_T \text{ and } J\mathfrak{D}_T \subset \mathfrak{D}_{T^*}.$$

Hence from the last two formulas we get

$$\mathfrak{D}_{T^*} = J(J\mathfrak{D}_{T^*}) \subset J\mathfrak{D}_T \subset \mathfrak{D}_{T^*}$$

and similarly

$$\mathfrak{D}_T = J(J\mathfrak{D}_T) \subset J\mathfrak{D}_{T^*} \subset \mathfrak{D}_T.$$

The formula

$$\begin{aligned} J_T D_T^2 &= J - T^*JT = J(J - TJT^*)J = JJ_{T^*}D_{T^*}^2 J = JJ_{T^*}JD_T^2 JJ \\ &= JJ_{T^*}JD_T^2 \end{aligned}$$

yields the equation (5.4).

The relation (5.5) follows from

$$D_T L_{T^*} = T^*JD_{T^*}\upharpoonright\mathfrak{D}_{T^*} = JTJD_TJ\upharpoonright\mathfrak{D}_{T^*} = JD_{T^*}L_TJ = D_TJL_TJ. \quad \square$$

5.2. Lemmas on negative indices of certain block operators

The first two lemmas are of preparatory nature for the last two lemmas, which are used for the proof of the main theorem.

Lemma 5.3. *Let* $\begin{pmatrix} J & T \\ T & J \end{pmatrix} : \begin{pmatrix} \mathfrak{H} \\ \mathfrak{H} \end{pmatrix} \to \begin{pmatrix} \mathfrak{H} \\ \mathfrak{H} \end{pmatrix}$ *be a selfadjoint operator in the Hilbert space* $\mathfrak{H}^2 = \mathfrak{H} \oplus \mathfrak{H}$. *Then*

$$\left|\begin{pmatrix} J & T \\ T & J \end{pmatrix}\right|^{1/2} = U \begin{pmatrix} |J+T|^{1/2} & 0 \\ 0 & |J-T|^{1/2} \end{pmatrix} U^*,$$

where $U = \frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ I & -I \end{pmatrix}$ *is a unitary operator.*

Proof. It is easy to check that

$$\begin{pmatrix} J & T \\ T & J \end{pmatrix} = U \begin{pmatrix} J+T & 0 \\ 0 & J-T \end{pmatrix} U^*. \tag{5.6}$$

Then by taking the modulus one gets

$$\left|\begin{pmatrix} J & T \\ T & J \end{pmatrix}\right|^2 = \left(\begin{pmatrix} J & T \\ T & J \end{pmatrix}^* \begin{pmatrix} J & T \\ T & J \end{pmatrix}\right) = U \begin{pmatrix} |J+T|^2 & 0 \\ 0 & |J-T|^2 \end{pmatrix} U^*.$$

The last step is to extract the square roots (twice) from the both sides of the equation:

$$\left|\begin{pmatrix} J & T \\ T & J \end{pmatrix}\right|^{1/2} = U \begin{pmatrix} |J+T|^{1/2} & 0 \\ 0 & |J-T|^{1/2} \end{pmatrix} U^*.$$

The right hand side can be written in this form because U is unitary. $\square$

Lemma 5.4. *Let* $T = T^* \in \mathfrak{H}$ *be a selfadjoint operator in a Hilbert space* $\mathfrak{H}$ *and let* $J = J^* = J^{-1}$ *be a fundamental symmetry in* $\mathfrak{H}$ *with* $\nu_-(J) < \infty$. *Then*

$$\nu_-(J - TJT) + \nu_-(J) = \nu_-(J - T) + \nu_-(J + T). \tag{5.7}$$

In particular, $\nu_-(J - TJT) < \infty$ *if and only if* $\nu_-(J \pm T) < \infty$.

Proof. Consider block operators $\begin{pmatrix} J & T \\ T & J \end{pmatrix}$ and $\begin{pmatrix} J+T & 0 \\ 0 & J-T \end{pmatrix}$. Equality (5.6) yields $\nu_-\begin{pmatrix} J & T \\ T & J \end{pmatrix} = \nu_-\begin{pmatrix} J+T & 0 \\ 0 & J-T \end{pmatrix}$. The negative index of $\begin{pmatrix} J+T & 0 \\ 0 & J-T \end{pmatrix}$ equals $\nu_-(J-T)+\nu_-(J+T)$ and the negative index of $\begin{pmatrix} J & T \\ T & J \end{pmatrix}$ is easy to find by using the equality

$$\begin{pmatrix} J & T \\ T & J \end{pmatrix} = \begin{pmatrix} I & 0 \\ TJ & I \end{pmatrix} \begin{pmatrix} J & 0 \\ 0 & J-TJT \end{pmatrix} \begin{pmatrix} I & JT \\ 0 & I \end{pmatrix}. \tag{5.8}$$

Then one gets (5.7). □

Let $(\mathfrak{H}_i, (J_i\cdot,\cdot))$ $(i=1,2)$ and $(\mathfrak{H}, (J\cdot,\cdot))$ be Pontryagin spaces, where $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ and $J = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix}$. Consider an operator $T_{11} = T_{11}^{[*]} \in [(\mathfrak{H}_1, J_1)]$ such that $\nu_-[I - T_{11}^2] = \kappa < \infty$; see (1.1). Denote $\widetilde{T}_{11} = J_1 T_{11}$, then $\widetilde{T}_{11} = \widetilde{T}_{11}^*$ in the Hilbert space $\mathfrak{H}_1$. Rewrite

$$\begin{aligned} \nu_-[I - T_{11}^2] &= \nu_-(J_1(I - T_{11}^2)) = \nu_-(J_1 - \widetilde{T}_{11} J_1 \widetilde{T}_{11}) \\ &= \nu_-((J_1 - \widetilde{T}_{11}) J_1 (J_1 + \widetilde{T}_{11})). \end{aligned}$$

Furthermore, denote

$$\begin{aligned} J_+ &= \operatorname{sign}(J_1(I - T_{11})) = \operatorname{sign}(J_1 - \widetilde{T}_{11}), \\ J_- &= \operatorname{sign}(J_1(I + T_{11})) = \operatorname{sign}(J_1 + \widetilde{T}_{11}), \\ J_{11} &= \operatorname{sign}(J_1(I - T_{11}^2)) \end{aligned} \tag{5.9}$$

and let $\kappa_+ = \nu_-[I - T_{11}]$ and $\kappa_- = \nu_-[I + T_{11}]$. Notice that $|I \mp T_{11}| = |J_1 \mp \widetilde{T}_{11}|$ and one has polar decompositions

$$I \mp T_{11} = J_1 J_\pm |I \mp T_{11}|. \tag{5.10}$$

Lemma 5.5. *Let* $T_{11} = T_{11}^{[*]} \in [(\mathfrak{H}_1, J_1)]$ *and* $T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \in [(\mathfrak{H}, J)]$ *be a selfadjoint extension of the operator* T_{11} *with* $\nu_-[I \pm T_{11}] < \infty$ *and* $\nu_-(J) < \infty$. *Then the following statements*

(i) $\nu_-[I \pm T_{11}] = \nu_-[I \pm T]$;

(ii) $\nu_-[I - T^2] = \nu_-[I - T_{11}^2] - \nu_-(J_2)$;

(iii) $\operatorname{ran} J_1 T_{21}^{[*]} \subset \operatorname{ran} |I \pm T_{11}|^{1/2}$

are connected by the implications (i) $\Leftrightarrow$ *(ii)* $\Rightarrow$ *(iii).*

Proof. The Lemma can be formulated in an equivalent way for the Hilbert space operators: the block operator $\widetilde{T} = JT = \begin{pmatrix} \widetilde{T}_{11} & \widetilde{T}_{12} \\ \widetilde{T}_{21} & \widetilde{T}_{22} \end{pmatrix}$ is a selfadjoint extension of $\widetilde{T}_{11} = \widetilde{T}_{11}^* \in [\mathfrak{H}_1]$. Then the following statements

(i') $\nu_-(J_1 \pm \widetilde{T}_{11}) = \nu_-(J \pm \widetilde{T})$

(ii') $\nu_-(J - \widetilde{T}J\widetilde{T}) = \nu_-(J_1 - \widetilde{T}_{11}J_1\widetilde{T}_{11}) - \nu_-(J_2)$;
(iii') $\operatorname{ran}\widetilde{T}_{12} \subset \operatorname{ran}|J_1 \pm \widetilde{T}_{11}|^{1/2}$

are connected by the implications $(i') \Leftrightarrow (ii') \Rightarrow (iii')$.

Hence it's sufficient to prove this form of the Lemma.

Let us prove the equivalence $(i') \Leftrightarrow (ii')$. Condition (ii') is equivalent to

$$\nu_-\begin{pmatrix} J_1 & \widetilde{T}_{11} \\ \widetilde{T}_{11} & J_1 \end{pmatrix} = \nu_-\begin{pmatrix} J & \widetilde{T} \\ \widetilde{T} & J \end{pmatrix}. \tag{5.11}$$

Indeed, in view of (5.8)

$$\nu_-\begin{pmatrix} J_1 & \widetilde{T}_{11} \\ \widetilde{T}_{11} & J_1 \end{pmatrix} = \nu_-(J_1) + \nu_-(J_1 - \widetilde{T}_{11}J_1\widetilde{T}_{11})$$

and

$$\nu_-\begin{pmatrix} J & \widetilde{T} \\ \widetilde{T} & J \end{pmatrix} = \nu_-(J) + \nu_-(J - \widetilde{T}J\widetilde{T}) = \nu_-(J_1) + \nu_-(J_2) + \nu_-(J - \widetilde{T}J\widetilde{T}).$$

By using Lemma 5.4, equality (5.11) is equivalent to

$$\nu_-(J_1 - \widetilde{T}_{11}) + \nu_-(J_1 + \widetilde{T}_{11}) = \nu_-(J - \widetilde{T}) + \nu_-(J + \widetilde{T}). \tag{5.12}$$

Hence, $(i') \Rightarrow (ii')$.

Because $\nu_-(J_1 \pm \widetilde{T}_{11}) \leq \nu_-(J \pm \widetilde{T})$, then (5.12) shows that $(ii') \Rightarrow (i')$.

Now we prove implication $(ii') \Rightarrow (iii')$;the arguments here will be useful also for the proof of Lemma 5.6 below. Use a permutation to transform the matrix in the right hand side of (5.11):

$$\nu_-\begin{pmatrix} J & \widetilde{T} \\ \widetilde{T} & J \end{pmatrix} = \nu_-\begin{pmatrix} J_1 & 0 & \widetilde{T}_{11} & \widetilde{T}_{12} \\ 0 & J_2 & \widetilde{T}_{21} & \widetilde{T}_{22} \\ \widetilde{T}_{11} & \widetilde{T}_{12} & J_1 & 0 \\ \widetilde{T}_{21} & \widetilde{T}_{22} & 0 & J_2 \end{pmatrix} = \nu_-\begin{pmatrix} J_1 & \widetilde{T}_{11} & 0 & \widetilde{T}_{12} \\ \widetilde{T}_{11} & J_1 & \widetilde{T}_{12} & 0 \\ 0 & \widetilde{T}_{21} & J_2 & \widetilde{T}_{22} \\ \widetilde{T}_{21} & 0 & \widetilde{T}_{22} & J_2 \end{pmatrix}.$$

Then condition (5.11) implies to the condition

$$\operatorname{ran}\begin{pmatrix} 0 & \widetilde{T}_{12} \\ \widetilde{T}_{12} & 0 \end{pmatrix} \subset \operatorname{ran}\left|\begin{pmatrix} J_1 & \widetilde{T}_{11} \\ \widetilde{T}_{11} & J_1 \end{pmatrix}\right|^{1/2};$$

(see Theorem 2.1). By Lemma 5.3 the last inclusion can be rewritten as

$$\operatorname{ran}\begin{pmatrix} 0 & \widetilde{T}_{12} \\ \widetilde{T}_{12} & 0 \end{pmatrix} \subset \operatorname{ran} U \begin{pmatrix} |J_1 + \widetilde{T}_{11}|^{1/2} & 0 \\ 0 & |J_1 - \widetilde{T}_{11}|^{1/2} \end{pmatrix} U^*,$$

where $U = \frac{1}{\sqrt{2}}\begin{pmatrix} I & I \\ I & -I \end{pmatrix}$ is a unitary operator. This inclusion is equivalent to

$$\begin{aligned} \operatorname{ran} U^* \begin{pmatrix} 0 & \widetilde{T}_{12} \\ \widetilde{T}_{12} & 0 \end{pmatrix} U &= \operatorname{ran} \begin{pmatrix} \widetilde{T}_{12} & 0 \\ 0 & -\widetilde{T}_{12} \end{pmatrix} \\ &\subset \operatorname{ran} \begin{pmatrix} |J_1 + \widetilde{T}_{11}|^{1/2} & 0 \\ 0 & |J_1 - \widetilde{T}_{11}|^{1/2} \end{pmatrix} \end{aligned}$$

and clearly this is equivalent to condition (iii').

Note that if $\widetilde{T}_{11}$ has a selfadjoint extension $\widetilde{T}$ satisfying (i'). Then by applying Theorem 2.1 (or [5, Theorem 1]) it yields (iii'). □

Lemma 5.6. *Let* $T_{11} = T_{11}^{[*]} \in [(\mathfrak{H}_1, J_1)]$ *be an operator and let*

$$T_1 = \begin{pmatrix} T_{11} \\ T_{21} \end{pmatrix} : (\mathfrak{H}_1, J_1) \to \begin{pmatrix} (\mathfrak{H}_1, J_1) \\ (\mathfrak{H}_2, J_2) \end{pmatrix}$$

be an extension of T_{11} *with* $\nu_-[I - T_{11}^2] < \infty$, $\nu_-(J_1) < \infty$, *and* $\nu_-(J_2) < \infty$. *Then for the conditions*

(i) $\nu_-[I_1 - T_{11}^2] = \nu_-[I_1 - T_1^{[*]} T_1] + \nu_-(J_2)$;
(ii) $\operatorname{ran} J_1 T_{21}^{[*]} \subset \operatorname{ran} |I - T_{11}^2|^{1/2}$;
(iii) $\operatorname{ran} J_1 T_{21}^{[*]} \subset \operatorname{ran} |I \pm T_{11}|^{1/2}$

the implications (i) ⇒ (ii) and (i) ⇒ (iii) hold.

Proof. First we prove that (i)⇒(ii). In fact, this follows from Theorem 3.1 by taking $A = I - T_{11}^2$ and $B = T_{21}$.

A proof of (i)⇒(iii) is quite similar to the proof used in Lemma 5.5. Statement (i) is equivalent the following equation:

$$\nu_- \begin{pmatrix} J_1 & \widetilde{T}_{11} \\ \widetilde{T}_{11} & J_1 \end{pmatrix} = \nu_- \begin{pmatrix} J & \widetilde{T}_1 \\ \widetilde{T}_1^* & J_1 \end{pmatrix}.$$

Indeed,

$$\begin{aligned} \nu_- \begin{pmatrix} J_1 & \widetilde{T}_{11} \\ \widetilde{T}_{11} & J_1 \end{pmatrix} &= \nu_- \begin{pmatrix} J_1 & 0 \\ 0 & J_1 - \widetilde{T}_{11} J_1 \widetilde{T}_{11} \end{pmatrix} \\ &= \nu_-(J_1 - \widetilde{T}_{11} J_1 \widetilde{T}_{11}) + \nu_-(J_1) < \infty \end{aligned}$$

and

$$\begin{aligned} \nu_- \begin{pmatrix} J & \widetilde{T}_1 \\ \widetilde{T}_1^* & J_1 \end{pmatrix} &= \nu_- \begin{pmatrix} J & 0 \\ 0 & J_1 - \widetilde{T}_1^* J \widetilde{T}_1 \end{pmatrix} \\ &= \nu_-(J_1 - \widetilde{T}_{11} J_1 \widetilde{T}_{11} - \widetilde{T}_{21}^* J_2 \widetilde{T}_{21}) + \nu_-(J_1) + \nu_-(J_2). \end{aligned}$$

Due to (i) the right hand sides coincide and then the left hand sides coincide as well.

Now let us permutate the matrix in the latter equation.

$$\nu_- \begin{pmatrix} J & \widetilde{T}_1 \\ \widetilde{T}_1^* & J_1 \end{pmatrix} = \nu_- \begin{pmatrix} J_1 & 0 & \widetilde{T}_{11} \\ 0 & J_2 & \widetilde{T}_{21} \\ \widetilde{T}_{11} & \widetilde{T}_{21}^* & J_1 \end{pmatrix} = \nu_- \begin{pmatrix} J_1 & \widetilde{T}_{11} & 0 \\ \widetilde{T}_{11} & J_1 & \widetilde{T}_{21}^* \\ 0 & \widetilde{T}_{21} & J_2 \end{pmatrix}.$$

It follows from [5, Theorem 1] that the condition (i) implies the condition

$$\begin{aligned} \operatorname{ran} \begin{pmatrix} 0 \\ \widetilde{T}_{21}^* \end{pmatrix} &\subset \operatorname{ran} \left| \begin{pmatrix} J_1 & \widetilde{T}_{11} \\ \widetilde{T}_{11} & J_1 \end{pmatrix} \right|^{1/2} \\ &= \operatorname{ran} U \begin{pmatrix} |J_1 + \widetilde{T}_{11}|^{1/2} & 0 \\ 0 & |J_1 - \widetilde{T}_{11}|^{1/2} \end{pmatrix} U^*, \end{aligned}$$

where $U = \frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ I & -I \end{pmatrix}$ is a unitary operator (see Lemma 5.3). Then, equivalently,

$$\operatorname{ran} \widetilde{T}_{21}^* \subset \operatorname{ran} |J_1 \pm \widetilde{T}_{11}|^{1/2}. \quad \square$$

5.3. Contractive extensions of contractions with minimal negative indices

Following to [5, 12, 14] we consider the problem of existence and a description of selfadjoint operators T in the Pontryagin space $\begin{pmatrix} (\mathfrak{H}_1, J_1) \\ (\mathfrak{H}_2, J_2) \end{pmatrix}$ such that $A_+ = I + T$ and $A_- = I - T$ solve the corresponding completion problems

$$A_\pm^0 = \begin{pmatrix} I \pm T_{11} & \pm T_{21}^{[*]} \\ \pm T_{21} & * \end{pmatrix}, \tag{5.13}$$

under *minimal index conditions* $\nu_-[I + T] = \nu_-[I + T_{11}]$, $\nu_-[I - T] = \nu_-[I - T_{11}]$, respectively. Observe, that by Lemma 5.5 the two minimal index conditions above are equivalent to single condition $\nu_-[I - T^2] = \nu_-[I - T_{11}^2] - \nu_-(J_2)$.

It is clear from Theorem 2.1 that the conditions $\operatorname{ran} J_1 T_{21}^{[*]} \subset \operatorname{ran} |I - T_{11}|^{1/2}$ and $\operatorname{ran} J_1 T_{21}^{[*]} \subset \operatorname{ran} |I + T_{11}|^{1/2}$ are necessary for the existence of solutions; however as noted already in [5] they are not sufficient even in the Hilbert space setting.

The next theorem gives a general solvability criterion for the completion problem (5.13) and describes all solutions to this problem. As in the definite case, there are minimal solutions A_+ and A_- which are connected to two extreme selfadjoint extensions T of

$$T_1 = \begin{pmatrix} T_{11} \\ T_{21} \end{pmatrix} : (\mathfrak{H}_1, J_1) \to \begin{pmatrix} (\mathfrak{H}_1, J_1) \\ (\mathfrak{H}_2, J_2) \end{pmatrix}, \tag{5.14}$$

now with finite negative index $\nu_-[I - T^2] = \nu_-[I - T_{11}^2] - \nu_-(J_2) > 0$. The set of solutions T to the problem (5.13) will be denoted by $\operatorname{Ext}_{T_1,\kappa}(-1,1)_{J_2}$.

Theorem 5.7. *Let* T_1 *be a symmetric operator as in* (5.14) *with* $T_{11} = T_{11}^{[*]} \in [(\mathfrak{H}_1, J_1)]$ *and* $\nu_-[I - T_{11}^2] = \kappa < \infty$, *and let* $J_{T_{11}} = \operatorname{sign}(J_1(I - T_{11}^2))$. *Then the* *completion problem for* $A^0_\pm$ *in* (5.13) *has a solution* $I \pm T$ *for some* $T = T^{[*]}$ *with* $\nu_-[I - T^2] = \kappa - \nu_-(J_2)$ *if and only if the following condition is satisfied:*

$$\nu_-[I - T_{11}^2] = \nu_-[I - T_1^{[*]} T_1] + \nu_-(J_2). \tag{5.15}$$

If this condition is satisfied then the following facts hold:

(i) *The completion problems for* $A^0_\pm$ *in* (5.13) *have "minimal" solutions* $A_\pm$ *(for the partial ordering introduced in the first section).*
(ii) *The operators* $T_m := A_+ - I$ *and* $T_M := I - A_- \in \operatorname{Ext}_{T_1,\kappa}(-1,1)_{J_2}$.
(iii) *The operators* T_m *and* T_M *have the block form*

$$\begin{aligned} T_m &= \begin{pmatrix} T_{11} & J_1 D_{T_{11}} V^* \\ J_2 V D_{T_{11}} & -I + J_2 V (I - L_T^* J_1) J_{11} V^* \end{pmatrix}, \\ T_M &= \begin{pmatrix} T_{11} & J_1 D_{T_{11}} V^* \\ J_2 V D_{T_{11}} & I - J_2 V (I + L_T^* J_1) J_{11} V^* \end{pmatrix}, \end{aligned} \tag{5.16}$$

where $D_{T_{11}} := |I - T_{11}^2|^{1/2}$ *and* V *is given by* $V := \operatorname{clos}(J_2 T_{21} D_{T_{11}}^{[-1]})$.
(iv) *The operators* T_m *and* T_M *are "extremal" extensions of* T_1*:*

$$T \in \operatorname{Ext}_{T_1,\kappa}(-1,1)_{J_2} \quad \textit{iff} \quad T = T^{[*]} \in [(\mathfrak{H}, J)], \quad T_m \leq_{J_2} T \leq_{J_2} T_M. \tag{5.17}$$

(v) *The operators* T_m *and* T_M *are connected via*

$$(-T)_m = -T_M, \quad (-T)_M = -T_m. \tag{5.18}$$

Proof. It is easy to see by (3.1) that $\kappa = \nu_-[I - T_{11}^2] \leq \nu_-[I - T_1^{[*]} T_1] + \nu_-(J_2) \leq \nu_-[I - T^2] + \nu_-(J_2)$. Hence the condition $\nu_-[I - T^2] = \kappa - \nu_-(J_2)$ implies (5.15). The sufficiency of this condition is obtained when proving the assertions (i)–(iii) below.

(i) If the condition (5.15) is satisfied then by using Lemma 5.6 one gets the inclusions $\operatorname{ran} J_1 T_{21}^{[*]} \subset \operatorname{ran} |I \pm T_{11}|^{1/2}$, which by Theorem 2.1 means that each of the completion problems, $A^0_\pm$ in (5.13), is solvable. It follows that the operators

$$S_- = |I + T_{11}|^{[-1/2]} J_1 T_{21}^{[*]}, \quad S_+ = |I - T_{11}|^{[-1/2]} J_1 T_{21}^{[*]} \tag{5.19}$$

are well defined and they provide the minimal solutions $A_\pm$ to the completion problems for $A^0_\pm$ in (5.13).

(ii) & (iii) By Lemma 5.6 the inclusion $\operatorname{ran} J_1 T_{21}^{[*]} \subset \operatorname{ran} |I - T_{11}^2|^{1/2}$ holds. This inclusion is equivalent to the existence of a (unique) bounded operator $V^* = D_{T_{11}}^{[-1]} J_1 T_{21}^{[*]}$ with $\ker V \supset \ker D_{T_{11}}$, such that $J_1 T_{21}^{[*]} = D_{T_{11}} V^*$. The operators $T_m := A_+ - I$ and $T_M := I - A_-$ (see proof of (i)) by using (5.1), (5.2), and 5.2 can be now rewritten as in (5.16). Indeed, observe that (see

Theorem 2.1, (5.9), and (5.10))

$$
\begin{aligned}
J_2S_-^*J_-S_- &= J_2VD_{T_{11}}|I+T_{11}|^{[-1/2]}J_-|I+T_{11}|^{[-1/2]}D_{T_{11}}V^* \\
&= J_2VD_{T_{11}}(J_1(I+T_{11}))^{[-1]}D_{T_{11}}V^* \\
&= J_2VD_{T_{11}}D_{T_{11}}^{[-1]}(I+L_{T_{11}}^*J_1)^{[-1]}D_{T_{11}}J_1D_{T_{11}}V^* \\
&= J_2V(I+L_{T_{11}}^*J_1)^{[-1]}(J_{11}-L_{T_{11}}^*J_{T_{11}^*}L_{T_{11}})V^* \\
&= J_2V(I+L_{T_{11}}^*J_1)^{[-1]}(J_{11}-(L_{T_{11}}^*J_1)^2J_{11})V^* \\
&= J_2V(I+L_{T_{11}}^*J_1)^{[-1]}(I+L_{T_{11}}^*J_1)(I-L_{T_{11}}^*J_1)J_{11}V^* \\
&= J_2V(I-L_{T_{11}}^*J_1)J_{11}V^*,
\end{aligned}
$$

where the third equality follows from (5.1) and the fourth from (5.2).

And similarly for

$$
\begin{aligned}
J_2S_+^*J_+S_+ &= J_2VD_{T_{11}}|I-T_{11}|^{[-1/2]}J_+|I-T_{11}|^{[-1/2]}D_{T_{11}}V^* \\
&= J_2VD_{T_{11}}(J_1(I-T_{11}))^{[-1]}D_{T_{11}}V^* \\
&= J_2VD_{T_{11}}D_{T_{11}}^{[-1]}(I-L_{T_{11}}^*J_1)^{[-1]}D_{T_{11}}J_1D_{T_{11}}V^* \\
&= J_2V(I-L_{T_{11}}^*J_1)^{[-1]}(J_{11}-L_{T_{11}}^*J_{T_{11}^*}L_{T_{11}})V^* \\
&= J_2V(I-L_{T_{11}}^*J_1)^{[-1]}(J_{11}-(L_{T_{11}}^*J_1)^2J_{11})V^* \\
&= J_2V(I-L_{T_{11}}^*J_1)^{[-1]}(I-L_{T_{11}}^*J_1)(I+L_{T_{11}}^*J_1)J_{11}V^* \\
&= J_2V(I+L_{T_{11}}^*J_1)J_{11}V^*,
\end{aligned}
$$

which implies the representations for T_m and T_M in (5.16). Clearly, T_m and T_M are selfadjoint extensions of T_1, which satisfy the equalities

$$\nu_-[I+T_m]=\kappa_-, \quad \nu_-[I-T_M]=\kappa_+.$$

Moreover, it follows from (5.16) that

$$T_M-T_m=\begin{pmatrix}0 & 0\\ 0 & 2(I-J_2VJ_{11}V^*)\end{pmatrix}. \tag{5.20}$$

Now the assumption (5.15) will be used again. Since $\nu_-[I-T_1^{[*]}T_1]=\nu_-[I-T_{11}^2]-\nu_-(J_2)$ and $T_{21}=J_2VD_{T_{11}}$ it follows from Theorem 3.1 that $V^*\in[\mathfrak{H}_2,\mathfrak{D}_{T_{11}}]$ is J-contractive: $J_2-VJ_{11}V^*\geq 0$. Therefore, (5.20) shows that $T_M\geq_{J_2}T_m$ and $I+T_M\geq_{J_2}I+T_m$ and hence, in addition to $I+T_m$, also $I+T_M$ is a solution to the problem A_+^0 and, in particular, $\nu_-[I+T_M]=\kappa_-=\nu_-[I+T_m]$. Similarly, $I-T_M\leq_{J_2}I-T_m$ which implies that $I-T_m$ is also a solution to the problem A_-^0, in particular, $\nu_-[I-T_m]=\kappa_+=\nu_-[I-T_M]$. Now by applying Lemma 5.5 we get

$$\nu_-[I-T_m^2]=\kappa-\nu_-(J_2),$$

$$\nu_-[I-T_M^2]=\kappa-\nu_-(J_2).$$

Therefore, $T_m,T_M\in \operatorname{Ext}_{T_1,\kappa}(-1,1)_{J_2}$ which in particular proves that the condition (5.15) is sufficient for solvability of the completion problem (5.13).

(iv) Observe, that $T \in \operatorname{Ext}_{T_1,\kappa}(-1,1)_{J_2}$ if and only if $T = T^{[*]} \supset T_1$ and $\nu_-[I \pm T] = \kappa_\mp$. By Theorem 2.1 this is equivalent to

$$J_2 S_-^* J_- S_- - I \leq_{J_2} T_{22} \leq_{J_2} I - J_2 S_+^* J_+ S_+. \tag{5.21}$$

The inequalities (5.21) are equivalent to (5.17).

(v) The relations (5.18) follow from (5.19) and (5.16). □

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References

[1] Antezana, J., Corach, G., and Stojanoff, D., Bilateral shorted operators and parallel sums. Linear Algebra Appl. **414** (2006), 570–588.

[2] Arsene, Gr. and Gheondea, A., Completing Matrix Contractions, J. Operator Theory, **7** (1982), 179–189.

[3] Arsene, Gr., Constantinescu, T., Gheondea, A., Lifting of Operators and Prescribed Numbers of Negative Squares, Michigan Math. J., **34** (1987), 201–216.

[4] Azizov, T.Ya. and Iokhvidov, I.S., *Linear operators in spaces with indefinite metric*, John Wiley and Sons, New York, 1989.

[5] Baidiuk, D., and Hassi, S., Completion, extension, factorization, and lifting of operators, Arxiv 2014, (to appear in Math. Ann.)

[6] Bognár, J., *Indefinite Inner Product Space*, Springer-Verlag, Berlin, 1974.

[7] Constantinescu, T. and Gheondea, A.: Minimal Signature of Lifting operators. I, J. Operator Theory, **22** (1989), 345–367.

[8] Constantinescu, T. and Gheondea, A.: Minimal Signature of Lifting operators. II, J. Funct. Anal., **103** (1992), 317–352.

[9] Davis, Ch., Kahan, W.M., and Weinberger, H.F., Norm preserving dilations and their applications to optimal error bounds, SIAM J. Numer. Anal., **19**, no. 3 (1982), 445–469.

[10] Dritschel, M.A., A lifting theorem for bicontractions on Kreĭn spaces, J. Funct. Anal., **89** (1990), 61–89.

[11] Dritschel, M.A. and Rovnyak, J., Extension theorems for contraction operators on Kreĭn spaces. *Extension and interpolation of linear operators and matrix functions*, 221–305, Oper. Theory Adv. Appl., **47**, Birkhäuser, Basel, 1990.

[12] Hassi, S., Malamud, M.M., and de Snoo, H.S.V., On Kreĭn's Extension Theory of Nonnegative Operators, Math. Nachr., **274/275** (2004), 40–73.

[13] Kato, T., *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin, Heidelberg, 1995.

[14] Kolmanovich, V.U. and Malamud, M.M., Extensions of Sectorial operators and dual pair of contractions, (Russian) Manuscript No 4428-85. Deposited at Vses. Nauchn-Issled, Inst. Nauchno-Techn. Informatsii, VINITI 19 04 85, Moscow, R ZH Mat 10B1144, (1985), 1–57.

[15] Kreĭn, M.G., On hermitian operators with defect indices (1, 1), Dokl. Akad. Nauk SSSR, **43** (1944), 339–342.

[16] Kreĭn, M.G., On resolvents of Hermitian operator with deficiency index (m, m), Dokl. Akad. Nauk SSSR, **52** (1946), 657–660.

[17] Kreĭn, M.G., Theory of Selfadjoint Extensions of Semibounded Operators and Its Applications, I, Mat. Sb. **20**, No.3 (1947), 431–498.

[18] Langer, H. and Textorius, B., Extensions of a bounded Hermitian operator T preserving the numbers of negative eigenvalues of $I - T^*T$, Research Report LiTH-MAT-R-87-17, Department of Mathematics, Linköping University, (1977), 15 pp.

[19] Shmul'yan, Yu. L., A Hellinger operator integral, (Russian) Mat. Sb. (N.S.), **49**, No.91 (1959), 381–430.

[20] Shmul'yan, Yu. L. and Yanovskaya, R.N., Blocks of a contractive operator matrix, Izv. Vyssh. Uchebn. Zaved. Mat., No. 7 (1981), 72–75.

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Boundary triplets and generalized resolvents of isometric operators in a Pontryagin space

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Abstract. The notions of the boundary triplet of an isometric operator V in the Pontryagin space and the corresponding function Weyl are introduced. Proper extensions of the isometric operator V, their spectra, and canonical and generalized resolvents of the operator V are described.

Keywords. Pontryagin space, boundary triplets of an isometric operator, Weyl function, canonical and generalized resolvents.

Introduction

The unitary operators in a space with indefinite metric and the problem of continuation of an isometric operator V were studied in works [3, 9, 10, 12]. In the case where the defect subspaces of an operator V are nondegenerate, the operator V is called standard, and the problem of its continuation causes no difficulties. The description of the generalized resolvents of a standard operator was given in [8]. For a nonstandard isometric operator, the description presented in [15, 16] is associated with significant technical difficulties related to the necessity to consider unitary linear relations in a Pontryagin space.

We propose another approach to the theory of the extensions of isometric operators in a Pontryagin space that is based on the notion of the boundary triplet of an isometric operator. In the case of a Hilbert space $\mathcal{H}$, this notion was introduced and applied to the classical problems of analysis in works by M. M. Malamud and V. I. Mogilevskii [13] and [14]. For a Pontryagin space, the definition of boundary triplet is a partial case of the definition of boundary relation in [4]. The advantage of our approach consists in that the auxiliary space in the definition of boundary triplet is a Hilbert one. Thus, the problem of continuation of the isometric operator V in a Pontryagin space can be solved simply, as in the case of a Hilbert space. Here, we introduce the notion of the Weyl function of an isometric operator, which generalizes the appropriate definition from [13] and study its properties. This will allow us to describe the properties of the extensions of the operator V, as well as the generalized resolvents of an isometric operator in a Pontryagin space.

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1. Preliminary information

1.1. Linear relations

We recall some information about linear relations from [5, 7]. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces. The linear relation (l.r.) T from $\mathcal{H}_1$ in $\mathcal{H}_2$ is a linear subspace in $\mathcal{H}_1 \times \mathcal{H}_2$. If the linear operator T is identified with its graph, then the set $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ of linear bounded operators from $\mathcal{H}_1$ to $\mathcal{H}_2$ is contained in the set of linear relations from $\mathcal{H}_1$ to $\mathcal{H}_2$. In what follows, we interpret the linear relation $T : \mathcal{H}_1 \to \mathcal{H}_2$ as a multivalued linear mapping from $\mathcal{H}_1$ to $\mathcal{H}_2$. If $\mathcal{H} := \mathcal{H}_1 = \mathcal{H}_2$, we say that T is a linear relation in $\mathcal{H}$.

For a linear relation $T : \mathcal{H}_1 \to \mathcal{H}_2$, we denote $\operatorname{dom} T$, $\ker T$, $\operatorname{ran} T$, and $\operatorname{mul} T$ as the domain of definition, kernel, range, and multivalued part, respectively. The inverse relation T^{-1} is a linear relation from $\mathcal{H}_2$ to $\mathcal{H}_1$ defined by the equality

$$T^{-1} = \left\{ \begin{bmatrix} f' \\ f \end{bmatrix} : \begin{bmatrix} f \\ f' \end{bmatrix} \in T \right\}.$$

The sum $T + S$ of two linear relations T and S is defined in the form

$$T + S = \left\{ \begin{bmatrix} f \\ g + h \end{bmatrix} : \begin{bmatrix} f \\ g \end{bmatrix} \in T, \ \begin{bmatrix} f \\ h \end{bmatrix} \in S \right\}. \tag{1.1}$$

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Banach spaces. By $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, we denote the set of all linear bounded operators from $\mathcal{H}_1$ to $\mathcal{H}_2$; $\mathcal{B}(\mathcal{H}) := \mathcal{B}(\mathcal{H}, \mathcal{H})$. We recall that the point $\lambda \in \mathbb{C}$ is called a point of the regular type of an operator $T \in \mathcal{B}(\mathcal{H})$, if there exists $c_\lambda > 0$ such that

$$\|(T - \lambda I)f\|_{\mathcal{H}} \geq c_\lambda \|f\|_{\mathcal{H}}, \quad f \in \mathcal{H}.$$

If $\operatorname{ran}(T - \lambda I) = \mathcal{H}$ in this case, then λ is called a regular point of the operator T. By $\rho(T)$ $(\widehat{\rho}(T))$, we denote the set of regular (regular type) points of the operator T.

1.2. Linear relations in Pontryagin spaces

Let $\mathcal{H}$ be a Hilbert space, and let $j_{\mathcal{H}}$ be a signature operator in it, i.e., $j_{\mathcal{H}} = j_{\mathcal{H}}^* = j_{\mathcal{H}}^{-1}$. We interpret the space $\mathcal{H}$ as a Krein space $(\mathcal{H}, j_{\mathcal{H}})$ (see [3]) in which the indefinite scalar product is defined by the equality $[\varphi, \psi]_{\mathcal{H}} = (j_{\mathcal{H}}\varphi, \psi)_{\mathcal{H}}$. The signature operator $j_{\mathcal{H}}$ can be presented in the form $j_{\mathcal{H}} = P_+ - P_-$, where P_+ and P_- are orthoprojectors in $\mathcal{H}$. In the case where P_- is finite-dimensional, and $\dim P_-\mathcal{H} = \kappa$, the Krein space $(\mathcal{H}, j_{\mathcal{H}})$ is called a Pontryagin space with negative index κ, which is denoted by $\operatorname{ind}_-\mathcal{H} = \kappa$.

Consider two Pontryagin spaces $(\mathcal{H}_1, j_{\mathcal{H}_1})$ and $(\mathcal{H}_2, j_{\mathcal{H}_2})$ and a linear relation T from $\mathcal{H}_1$ to $\mathcal{H}_2$. Then the adjoint linear relation $T^{[*]}$ consists of pairs $\left[\begin{smallmatrix} g_2 \\ g_1 \end{smallmatrix}\right] \in \mathcal{H}_2 \times \mathcal{H}_1$ such that

$$[f_2, g_2]_{\mathcal{H}_2} = [f_1, g_1]_{\mathcal{H}_1}, \quad \text{for all } \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \in T.$$

If T^* is the l.r. adjoint to T considered as a l.r. from the Hilbert space $\mathcal{H}_1$ to the Hilbert space $\mathcal{H}_2$, then $T^{[*]} = j_{\mathcal{H}_1} T^* j_{\mathcal{H}_2}$.

The l.r. $T^{[*]}$ satisfies the equalities

$$(\operatorname{dom} T)^{[\perp]} = \operatorname{mul} T^{[*]}, \quad (\operatorname{ran} T)^{[\perp]} = \ker T^{[*]}, \tag{1.2}$$

where the sign $[\perp]$ means the orthogonality in a Pontryagin space.

Definition 1.1. *A linear relation T from a Pontryagin space $(\mathcal{H}_1, j_{\mathcal{H}_1})$ to a Pontryagin space $(\mathcal{H}_2, j_{\mathcal{H}_2})$ is called isometric, if, for all $\left[\begin{smallmatrix}\varphi\\ \varphi'\end{smallmatrix}\right] \in T$, the equality*

$$[\varphi', \varphi']_{\mathcal{H}_2} = [\varphi, \varphi]_{\mathcal{H}_1} \tag{1.3}$$

holds. It is called a contractive (expanding) one, if equality (1.3) is replaced by an inequality with the sign $\leq$ ($\geq$). A linear relation from $(\mathcal{H}_1, j_{\mathcal{H}_1})$ to $(\mathcal{H}_2, j_{\mathcal{H}_2})$ is called unitary, if $T^{-1} = T^{[*]}$. These properties are invariant relative to the closure. It is easy to obtain from (1.3) that a linear relation T is isometric iff $T^{-1} \subset T^{[*]}$.

As is known [3], the sets $\mathbb{D} \setminus \hat{\rho}(T)$ and $\mathbb{D}_e \setminus \hat{\rho}(T)$ for an isometric operator T in a Pontryagin space with $\operatorname{ind}_- \mathcal{H} = \kappa$ consist of at most κ points, which belong to $\sigma_p(T)$.

The definition of unitary relation was first given in [17], where the following assertion was proved.

Proposition 1.1. *If T is a unitary relation, then*

1. $\operatorname{dom} T$ *is closed iff* $\operatorname{ran} T$ *is closed;*
2. *the equalities* $\ker T = \operatorname{dom} T^{[\perp]}$, $\operatorname{mul} T = \operatorname{ran} T^{[\perp]}$ *hold.*

Proposition 1.1 yields the following result.

Corollary 1.1. *If T is a unitary relation in a Pontryagin space, then* $\operatorname{mul} T \neq \{0\}$ *if and only if* $\ker T \neq \{0\}$. *In this case,* $\dim \operatorname{mul} T = \dim \ker T$.

2. Boundary triplets for an isometric operator in a Pontryagin space

2.1. Boundary triplets and description of the extensions of an isometric operator in a Pontryagin space

In the case where $\mathcal{H}$ is a Hilbert space, the definition of the boundary triplet for an isometric operator was introduced in [13]. We note that the notion of the boundary triplet of an isometric operator, which will be introduced below in definition 2.1, is a partial case of the notion of the boundary relation of an isometric operator in a Pontryagin space [4].

Let $\mathcal{H}$ be a Pontryagin space with negative index κ, and let the operator $V : \mathcal{H} \to \mathcal{H}$ be an isometry in $\mathcal{H}$. By $\mathfrak{N}_1$ and $\mathfrak{N}_2$, we denote two auxiliary Hilbert spaces.

Definition 2.1. *The collection $\Pi = \{\mathfrak{N}_1 \oplus \mathfrak{N}_2, \Gamma_1, \Gamma_2\}$ is called the boundary triplet of an isometric operator V, if*

1) *the following Green's generalized identity holds:*

$$[f', g']_{\mathcal{H}} - [f, g]_{\mathcal{H}} = (\Gamma_1 \widehat{f}, \Gamma_1 \widehat{g})_{\mathfrak{N}_1} - (\Gamma_2 \widehat{f}, \Gamma_2 \widehat{g})_{\mathfrak{N}_2}, \tag{2.1}$$

where $\widehat{f} = \begin{bmatrix} f \\ f' \end{bmatrix}$, $\widehat{g} = \left[\begin{smallmatrix} g \\ g' \end{smallmatrix}\right] \in V^{-[*]}$;

2) *the mapping* $\Gamma = (\Gamma_1, \Gamma_2)^T : V^{-[*]} \to \mathfrak{N}_1 \oplus \mathfrak{N}_2$ *is surjective.*

For an isometric operator, it is convenient to define the defect subspace $\mathfrak{N}_\lambda(V)$ as follows:

$$\mathfrak{N}_\lambda(V) := \ker \left(I - \lambda V^{[*]}\right) = \left\{ f_\lambda : \begin{bmatrix} f_\lambda \\ \lambda f_\lambda \end{bmatrix} \in V^{-[*]} \right\}. \tag{2.2}$$

For $\lambda \in \widehat{\rho}(V)$, $\mathfrak{N}_\lambda(V)$ is a closed subspace in $\mathcal{H}$ [3].

We also set

$$\widehat{\mathfrak{N}}_\lambda(V) := \left\{ \begin{bmatrix} f_\lambda \\ \lambda f_\lambda \end{bmatrix} : f_\lambda \in \mathfrak{N}_\lambda(V) \right\}. \tag{2.3}$$

It follows from (2.2) that $\widehat{\mathfrak{N}}_\lambda(V) \subset V^{-[*]}$.

Proposition 2.1. *For any isometric operator $V : \mathcal{H} \to \mathcal{H}$, where $\mathcal{H}$ is a Pontryagin space Π_κ, there exists a boundary triplet.*

Proof. Since V is an isometric operator, it is a neutral subspace of the space $\mathcal{H}^2$ with indefinite scalar product

$$[\widehat{f}, \widehat{f}]_{\mathcal{H}^2} := (J_{\mathcal{H}} \widehat{f}, \widehat{f})_{\mathcal{H}^2} = [f, f]_{\mathcal{H}} - [f', f']_{\mathcal{H}}, \quad \text{where } J_{\mathcal{H}} = \begin{bmatrix} I_{\mathcal{H}} & 0 \\ 0 & -I_{\mathcal{H}} \end{bmatrix}.$$

Let us identify V with its graph in $\mathcal{H}^2$. Then the lineal $V^{-[*]}$ that is orthogonal to V can be presented in the form ([3, p. 44])

$$V^{-[*]} = V[\dot{+}]\mathfrak{D}_+[\dot{+}]\mathfrak{D}_-, \tag{2.4}$$

where $\mathfrak{D}_+$ and $\mathfrak{D}_-$ are some positive and negative subspaces in $(\mathcal{H}^2, J_{\mathcal{H}})$. For two arbitrary vectors $\widehat{f}$ and $\widehat{g}$ from $V^{-[*]}$, we consider the decompositions corresponding to (2.4):

$$\widehat{f} = \widehat{f}_0 + \widehat{u}_+ + \widehat{u}_-, \quad \widehat{g} = \widehat{g}_0 + \widehat{v}_+ + \widehat{v}_-, \quad \text{where } \widehat{f}_0, \widehat{g}_0 \in V \text{ and } \widehat{u}_\pm, \widehat{v}_\pm \in \mathfrak{D}_\pm.$$

We define boundary operators as follows:

$$\Gamma_1 \widehat{f} = \widehat{u}_+, \Gamma_2 \widehat{f} = \widehat{v}_-.$$

The spaces $(\mathfrak{D}_+, J_{\mathcal{H}})$ and $(\mathfrak{D}_-, -J_{\mathcal{H}})$ are Hilbert ones. Then the collection $\Pi = \{\mathfrak{D}_+ \oplus \mathfrak{D}_-, \Gamma_1, \Gamma_2\}$ is the boundary triplet for the isometric operator V, since

$$\begin{aligned}[\widehat{f}, \widehat{g}]_{\mathcal{H}^2} &= [\widehat{f}_0, \widehat{g}_0]_{\mathcal{H}^2} + [\widehat{u}_+, \widehat{v}_+]_{\mathcal{H}^2} + [\widehat{u}_-, \widehat{v}_-]_{\mathcal{H}^2} \\ &= (\widehat{u}_+, \widehat{v}_+)_{\mathfrak{D}_+} - (\widehat{u}_-, \widehat{v}_-)_{\mathfrak{D}_-} = (\Gamma_1 \widehat{f}, \Gamma_1 \widehat{g})_{\mathfrak{D}_+} - (\Gamma_2 \widehat{f}, \Gamma_2 \widehat{g})_{\mathfrak{D}_-}.\end{aligned}$$

The surjectivity of the mapping $\Gamma = \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix}$ is obvious. □

Let θ be a linear relation from $\mathfrak{N}_2$ in $\mathfrak{N}_1$. We define the extension V_θ of the operator V by the equality

$$V_\theta = \left\{ \widehat{f} \in V^{-[*]} : \begin{bmatrix} \Gamma_2 \widehat{f} \\ \Gamma_1 \widehat{f} \end{bmatrix} \in \theta \right\}. \tag{2.5}$$

The extension V_θ is, generally speaking, a linear relation from $\mathcal{H}$ to $\mathcal{H}$.

We define two extensions V_1 and V_2 of the operator V:

$$V_i = \left\{ \widehat{f} \in V^{-[*]} : \Gamma_i \widehat{f} = 0 \right\}, \quad i = 1, 2. \tag{2.6}$$

We note also that

$$V = \left\{ \widehat{f} \in V^{-[*]} : \Gamma_1 \widehat{f} = 0 \text{ and } \Gamma_2 \widehat{f} = 0 \right\}. \tag{2.7}$$

We now define two sets of points:

$$\Lambda_1 = \{\lambda \in \mathbb{D}_e : \widehat{\mathfrak{N}}_\lambda(V) \cap V_1 \neq \{0\}\} = \sigma_p(V_1) \cap \mathbb{D}_e; \tag{2.8}$$

$$\Lambda_2 = \{\lambda \in \mathbb{D} : \widehat{\mathfrak{N}}_\lambda(V) \cap V_2 \neq \{0\}\} = \sigma_p(V_2) \cap \mathbb{D}. \tag{2.9}$$

It will be proved in Lemma 2.1 that the extension V_1 is contractive in $\mathcal{H}$, whereas V_2 is an expanding operator in $\mathcal{H}$. As is known ([3, p. 186]), the spectrum of the contractive operator V_1 contains at most κ points in $\mathbb{D}_e$, and the spectrum of the expanding operator V_2 contains at most κ points in $\mathbb{D}$. Thus, each of the sets Λ_1 and Λ_2 contains at most κ points, and sets $\mathcal{D}_1 := \mathbb{D}_e \setminus \Lambda_1$ and $\mathcal{D}_2 := \mathbb{D} \setminus \Lambda_2$ are contained in the sets of regular points of these extensions.

Lemma 2.1. *Let the collection* $\Pi = \{\mathfrak{N}_1 \oplus \mathfrak{N}_2, \Gamma_1, \Gamma_2\}$ *be the boundary triplet of the isometric operator* V. *Then*

1) *the extension* V_1 *is contractive in* $\mathcal{H}$, *and* V_2 *is expanding in* $\mathcal{H}$;
2) *for all* $\lambda \in \mathcal{D}_1 = \rho(V_1) \cap \mathbb{D}_e$,
$$V^{-[*]} = V_1 \dotplus \widehat{\mathfrak{N}}_\lambda(V); \tag{2.10}$$
3) *for all* $\lambda \in \mathcal{D}_2 = \rho(V_2) \cap \mathbb{D}$,
$$V^{-[*]} = V_2 \dotplus \widehat{\mathfrak{N}}_\lambda(V). \tag{2.11}$$

Proof. 1) For all vectors $\widehat{f} \in V_1$, identity (2.1) and the definition of the l.r. V_1 (2.6) yield

$$[f', f'] - [f, f] = -\left(\Gamma_2 \widehat{f}, \Gamma_2 \widehat{f}\right)_{\mathfrak{N}_2} \leq 0$$

i.e., $[f', f'] \leq [f, f]$.

In the same way, we obtain the inverse inequality for V_2.

2) and 3). We now prove equality (2.10) (equality (2.11) can be proved analogously). For this purpose, we set the inclusion $V^{-[*]} \subset V_1 \dotplus \widehat{\mathfrak{N}}_\lambda(V)$. Consider a pair of vectors $\begin{bmatrix} f \\ f' \end{bmatrix} \in V^{-[*]}$. Let $f_1 = (V_1 - \lambda)^{-1}(f' - \lambda f)$ be a solution of the equation

$$f' - \lambda f = f'_1 - \lambda f_1, \quad \text{where } \begin{bmatrix} f_1 \\ f'_1 \end{bmatrix} \in V_1,$$

which is determined uniquely for $\lambda \in \mathcal{D}_1$. Then $f' - f'_1 = \lambda(f - f_1)$, i.e., $\begin{bmatrix} f - f_1 \\ \lambda(f - f_1) \end{bmatrix} \in V^{-[*]}$ and, hence, $f - f_1 \in \mathfrak{N}_\lambda(V)$. Since the inverse inclusion is obvious, equality (2.10) is proved. □

The following theorem will give description of proper extensions of the operator V, i.e., such that $V \subset V_\theta \subset V^{-[*]}$.

Theorem 2.1. *Let the collections* $\Pi = \{\mathfrak{N}_1 \oplus \mathfrak{N}_2, \Gamma_1, \Gamma_2\}$ *be the boundary triplet for* V, *let* θ *be a linear relation from* $\mathfrak{N}_2$ *to* $\mathfrak{N}_1$, *and let* V_θ *be the corresponding extension of the operator* V. *Then*

1) *the inclusion* $V_{\theta_1} \subset V_{\theta_2}$ *is equivalent to the inclusion* $\theta_1 \subset \theta_2$;
2) $V_{\theta^{-*}} = V_\theta^{-[*]}$;
3) V_θ *is a unitary extension of the operator* V, *iff* θ *is the graph of a unitary operator from* $\mathfrak{N}_2$ *to* $\mathfrak{N}_1$;
4) V_θ *is an isometric extension of the operator* V, *iff* θ *is the graph of an isometric operator from* $\mathfrak{N}_2$ *to* $\mathfrak{N}_1$;

5) V_θ *is a coisometric extension of the operator* V, *iff* θ *is the graph of a coisometric operator from* $\mathfrak{N}_2$ *to* $\mathfrak{N}_1$*;*

6) V_θ *is a contraction, iff* θ *is a contraction;*

7) V_θ *is an extension, iff* θ *is an extension.*

Proof. Assertion 1) follows obviously from the definition of V_{θ_1} and V_{θ_2}.

2) We take $\left[\begin{smallmatrix} f \\ f' \end{smallmatrix}\right] \in V_\theta$ and $\left[\begin{smallmatrix} g \\ g' \end{smallmatrix}\right] \in V_\theta^{-[*]}$. Then $\left[\begin{smallmatrix} g \\ g' \end{smallmatrix}\right] \in V_\theta^{[*]}$. From (2.1), we obtain

$$0 = [f', g'] - [f, g] = \left(\Gamma_1 \widehat{f}, \Gamma_1 \widehat{g}\right)_{\mathfrak{N}_1} - \left(\Gamma_2 \widehat{f}, \Gamma_2 \widehat{g}\right)_{\mathfrak{N}_2}$$

Since $\left[\begin{smallmatrix} \Gamma_2 \widehat{f} \\ \Gamma_1 \widehat{f} \end{smallmatrix}\right] \in \theta$, we have $\left[\begin{smallmatrix} \Gamma_1 \widehat{g} \\ \Gamma_2 \widehat{g} \end{smallmatrix}\right] \in \theta^*$ or $\left[\begin{smallmatrix} \Gamma_2 \widehat{g} \\ \Gamma_1 \widehat{g} \end{smallmatrix}\right] \in \theta^{-*}$, which means $\widehat{g} \in V_{\theta^{-*}}$. Hence, we show that $V_\theta^{-[*]} \subset V_{\theta^{-*}}$.

The inverse assertion can be proved by inversion of the above reasoning.

3) Let $V_\theta^{-[*]} = V_\theta$, i.e., let V_θ be a unitary extension of the operator V. Using the first assertion of this lemma, we obtain $\theta^{-*} = \theta$. Conversely, we set $\theta^{-*} = \theta$ and, by the first assertion of the lemma, arrive at $V_\theta^{-[*]} = V_\theta$.

4) and 5) are proved analogously. Assume that V_θ is a coisometry, i.e., $V_\theta^{-1} \supset V_\theta^{[*]}$. Then, by virtue of item 2), $V_{\theta^{-*}} = V_\theta^{-[*]} \subset V_\theta$. By virtue of assertion 1), we obtain $\theta^{-*} \subset \theta$, i.e., θ is a coisometry.

6) Let V_θ be a contraction. Then,for $\widehat{f} = \left[\begin{smallmatrix} f \\ f' \end{smallmatrix}\right] \in V_\theta$, formula (2.1) yields

$$0 \geq [f', f'] - [f, f] = \left(\Gamma_1 \widehat{f}, \Gamma_1 \widehat{f}\right)_{\mathfrak{N}_1} - \left(\Gamma_2 \widehat{f}, \Gamma_2 \widehat{f}\right)_{\mathfrak{N}_2}.$$

We obtain $(\Gamma_1 \widehat{f}, \Gamma_1 \widehat{g})_{\mathfrak{N}_1} \leq (\Gamma_2 \widehat{f}, \Gamma_2 \widehat{g})_{\mathfrak{N}_2}$. This means that θ is a contraction.

7) is proved analogously to 6). □

Remark 2.1. In assertions (3)–(6), the extension V_θ can be a a linear relation with nontrivial multi-valued part, whereas θ is the graph of a univalent operator.

Example 2.1. Let $\mathcal{H} = \mathbb{C}^2$, $J = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$. We take $V = \{0\}$. Then, for

$$V^{-[*]} = \left\{ \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} \right\} : x_1, x_2, x_1', x_2' \in \mathbb{C} \right\},$$

expansion (2.4), where

$$\mathfrak{D}_+ = \left\{ \begin{bmatrix} x_1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ x_2' \end{bmatrix} \right\}, \quad \mathfrak{D}_- = \left\{ \begin{bmatrix} 0 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1' \\ 0 \end{bmatrix} \right\},$$

is valid. In this case, the boundary operators can be defined by the equalities

$$\Gamma_1 \widehat{x} = P_+ \widehat{x} = \left\{ \begin{bmatrix} x_1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ x_2' \end{bmatrix} \right\} \sim \begin{bmatrix} x_1 \\ x_2' \end{bmatrix} \in \mathbb{C}^2,$$

$$\Gamma_2 \widehat{x} = P_- \widehat{x} = \left\{ \begin{bmatrix} 0 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1' \\ 0 \end{bmatrix} \right\} \sim \begin{bmatrix} x_1' \\ x_2 \end{bmatrix} \in \mathbb{C}^2.$$

As a unitary extension $\widetilde{V}$ of the operator V, we take the linear relation

$$\widetilde{V} = \left\{ \left\{ \begin{bmatrix} x_1 \\ x_1 \end{bmatrix}, \begin{bmatrix} x_1' \\ x_1' \end{bmatrix} \right\} : x_1, x_1' \in \mathbb{C} \right\}$$

for which $\operatorname{dom} \widetilde{V}$ and $\operatorname{ran} \widetilde{V}$ are neutral subspaces. Then, for all $\widehat{x} \in \widehat{V}$, we obtain $\Gamma_1 \widehat{x} = \Gamma_2 \widehat{x} = \begin{bmatrix} x_1 \\ x_1' \end{bmatrix}$. Hence, $\theta = \{\{\Gamma_2 \widehat{x}, \Gamma_1 \widehat{x}\} : \widehat{x} \in \widetilde{V}\}$ is the identity operator from $\mathbb{C}^2$ in $\mathbb{C}^2$, whereas $\widetilde{V}$ is not an operator.

2.2. γ-field and Weyl function

The notion of the Weyl function of an isometric operator V in a Hilbert space, which allows one to describe the analytic properties of extensions of the operator V, was introduced in [13]. In this section, we will generalize this notion to the case of the isometric operator V in a Pontryagin space.

Lemma 2.2. *Let $\Pi = \{\mathfrak{N}_1 \oplus \mathfrak{N}_2, \Gamma_1, \Gamma_2\}$ be the boundary triplet for V, and let V_1 and V_2 be the extensions of the isometric operator V that are defined in* (2.6). *Then the mappings $\Gamma_j \upharpoonright \widehat{\mathfrak{N}}_\lambda(V) : \widehat{\mathfrak{N}}_\lambda(V) \to \mathfrak{N}_j$, $j = 1, 2$, are bounded and boundedly invertible for $\lambda \in \mathcal{D}_j$.*

In this case, the operator-functions

$$\gamma_j(\lambda) := \pi_1 \widehat{\gamma}_j(\lambda) = \pi_1 \left(\Gamma_j \upharpoonright \widehat{\mathfrak{N}}_\lambda(V) \right)^{-1} \tag{2.12}$$

satisfy the equality

$$\gamma_j(\lambda) = (I + (\lambda - \mu)(V_j - \lambda)^{-1})\gamma_j(\mu), \quad \text{for} \quad \lambda, \mu \in \mathcal{D}_j, \quad j = 1, 2. \tag{2.13}$$

The operator-functions $\gamma_j(\cdot)$ are called γ-fields for the l.r. $V^{-[*]}$.

Proof. First, we will show that the mapping $\Gamma : V^{-[*]} \to \begin{bmatrix} \mathfrak{N}_1 \\ \mathfrak{N}_2 \end{bmatrix}$ is closed. Let $\widehat{f}_n = \begin{bmatrix} f_n \\ f_n' \end{bmatrix} \in V^{-[*]}$ and $\widehat{f}_n \to 0$. Then $\Gamma \widehat{f}_n = \begin{bmatrix} \Gamma_1 \widehat{f}_n \\ \Gamma_2 \widehat{f}_n \end{bmatrix} \to \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} =: h$. From (2.1), we obtain $(h_1, \Gamma_1 \widehat{g}) - (h_2, \Gamma_2 \widehat{g}) = 0$. The surjectivity of Γ implies that there exists a vector $g \in \mathcal{H}$ such that $\Gamma \widehat{g} = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$. From the previous equality, we obtain $\|h_1\|^2 + \|h_2\|^2 = 0$; hence, $h = 0$.

Since $\operatorname{dom} \Gamma = V^{-[*]}$, the operator Γ is bounded by the Banach theorem of closed graph. Hence, Γ_1 and Γ_2 are bounded as well.

By virtue of equality (2.10) and the surjectivity of Γ, the mapping $\Gamma_1 \upharpoonright \widehat{\mathfrak{N}}_\lambda(V) : \widehat{\mathfrak{N}}_\lambda(V) \to \mathfrak{N}_1$ acts on all $\mathfrak{N}_1$. From (2.1), we obtain the following estimate:

$$\|\Gamma_1 \widehat{f}_\lambda\|^2_{\mathfrak{N}_1} = \|\Gamma_2 \widehat{f}_\lambda\|^2_{\mathfrak{N}_2} + (|\lambda|^2 - 1)[f_\lambda, f_\lambda] \geq (|\lambda|^2 - 1)[f_\lambda, f_\lambda], \quad \text{where } \lambda \in \mathcal{D}_1, \ \widehat{f}_\lambda = \begin{bmatrix} f_\lambda \\ \lambda f_\lambda \end{bmatrix} \in \widehat{\mathfrak{N}}_\lambda(V).$$

Hence, $\Gamma_1 \upharpoonright \widehat{\mathfrak{N}}_\lambda(V)$ is boundedly invertible.

Analogously, we can prove the bounded invertibility of $\Gamma_2 \upharpoonright \widehat{\mathfrak{N}}_\lambda(V)$. Hence, $\gamma_j(\lambda)$ for $\lambda \in \mathcal{D}_j$ $(j = 1, 2)$ are defined properly.

We now prove identity (2.13). For definiteness, we take $j = 1$ and will prove that

$$\gamma_1(\lambda) = (I + (\lambda - \mu)(V_1 - \lambda)^{-1})\gamma_1(\mu), \quad \text{for} \quad \lambda, \mu \in \mathcal{D}_1.$$

Consider the vector $g_\mu = \gamma_1(\mu) h_1 \in \mathfrak{N}_\mu(V)$, where $h_1 \in \mathfrak{N}_1$. Then there exists the vector $h_2 \in \mathfrak{N}_2$ such that $\Gamma \begin{bmatrix} g_\mu \\ \mu g_\mu \end{bmatrix} = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$. We set

$$f_\lambda = g_\mu + (\lambda - \mu)(V_1 - \lambda)^{-1} g_\mu.$$

Then

$$\widehat{f}_\lambda = \begin{bmatrix} g_\mu \\ \mu g_\mu \end{bmatrix} + (\lambda-\mu)\begin{bmatrix} (V_1-\lambda)^{-1}g_\mu \\ \left(I+\lambda(V_1-\lambda)^{-1}\right)g_\mu \end{bmatrix}. \tag{2.14}$$

In this equality,

$$\widehat{g}_\mu = \begin{bmatrix} g_\mu \\ \mu g_\mu \end{bmatrix} \in \widehat{\mathfrak{N}}_\mu(V) \subset V^{-[*]}, \quad \begin{bmatrix} (V_1-\lambda)^{-1} \\ I+\lambda(V_1-\lambda)^{-1} \end{bmatrix} g_\mu \in V_1 \subset V^{-[*]}.$$

Thus, $\widehat{f}_\lambda \in \widehat{\mathfrak{N}}_\lambda(V)$.

Below, we will use an equality that follows from (2.6):

$$\Gamma\begin{bmatrix} (V_1-\lambda)^{-1}g_\mu \\ \left(I+\lambda(V_1-\lambda)^{-1}\right)g_\mu \end{bmatrix} = \begin{bmatrix} 0 \\ h_2' \end{bmatrix}. \tag{2.15}$$

Equalities (2.14) and (2.15) yield

$$\Gamma\widehat{f}_\lambda = \Gamma\begin{bmatrix} g_\mu \\ \mu g_\mu \end{bmatrix} + (\lambda-\mu)\Gamma\begin{bmatrix} (V_1-\lambda)^{-1}g_\mu \\ \left(I+\lambda(V_1-\lambda)^{-1}\right)g_\mu \end{bmatrix} = \begin{bmatrix} h_1 \\ h_2+(\lambda-\mu)h_2' \end{bmatrix}.$$

Hence, $f_\lambda = \gamma_1(\lambda)h_1$. This proves (2.13). □

The previous lemma implies that it is possible to define the operator-functions $M_1(\cdot)$ and $M_2(\cdot)$:

$$M_1(\lambda)\Gamma_1 \upharpoonright \mathfrak{N}_\lambda(V) = \Gamma_2 \upharpoonright \mathfrak{N}_\lambda(V), \quad \lambda \in \mathcal{D}_1; \tag{2.16}$$

$$M_2(\lambda)\Gamma_2 \upharpoonright \mathfrak{N}_\lambda(V) = \Gamma_1 \upharpoonright \mathfrak{N}_\lambda(V), \quad \lambda \in \mathcal{D}_2. \tag{2.17}$$

It follows from definition (2.12) of $\gamma_1(\cdot)$ and $\gamma_2(\cdot)$ that $M_1(\lambda)$ and $M_2(\lambda)$ are defined properly, and

$$M_1(\lambda) := \Gamma_2\widehat{\gamma}_1(\lambda), \quad \lambda \in \mathcal{D}_1; \tag{2.18}$$

$$M_2(\lambda) := \Gamma_1\widehat{\gamma}_2(\lambda), \quad \lambda \in \mathcal{D}_2. \tag{2.19}$$

In what follows, we need the Shur class S and the generalized Shur class S_κ of functions. Their definition is given below.

Definition 2.2. *A function $s(\lambda)$ defined and holomorphic in a domain $\mathfrak{h}_s \subset \mathbb{D}$ belongs to the class $S_\kappa(\mathfrak{N}_1,\mathfrak{N}_2)$, if the kernel $K_\mu(\lambda) = \frac{1-s(\mu)^*s(\lambda)}{1-\lambda\overline{\mu}}$ has κ negative squares, i.e., for all $\lambda_1,\dots,\lambda_n \in \mathfrak{h}_s$ and $u_1,\dots,u_n \in \mathfrak{N}_1$, the matrix $((K_{\lambda_j}(\lambda_i)u_i,u_j))_{i,j=1}^n$ has at most κ negative eigenvalues. For at least one such choice, it has exactly κ negative eigenvalues.*

In particular, an $[\mathfrak{N}_1,\mathfrak{N}_2]$-valued function $s(\cdot)$ belongs to the class $S(\mathfrak{N}_1,\mathfrak{N}_2)$, if the kernel $K_\mu(\lambda)$ is positive definite everywhere in $\mathbb{D}$. As is known, the last condition is equivalent to that $s(\cdot)$ is holomorphic in $\mathbb{D}$, and $\|s(\lambda)\| \le 1$ for all $\lambda \in \mathbb{D}$.

Proposition 2.2. *The operator-function $M_2(\cdot)$ belongs to $S_\kappa(\mathfrak{N}_2,\mathfrak{N}_1)$.*

Proof. Let λ_j be some points from $\mathcal{D}_2$, $j=1,\dots,n$. We denote $h_j := \Gamma_2\widehat{f}_{\lambda_j}$. Then $\Gamma_1\widehat{f}_{\lambda_j} = M_2(\lambda)h_j$. From (2.1) for $\widehat{f}_{\lambda_j}$ and $\widehat{f}_{\lambda_k}$, we have

$$(\lambda_j\overline{\lambda}_k - 1)[f_{\lambda_j}, f_{\lambda_k}] = (M_2(\lambda_j)h_j, M_2(\lambda_k)h_k)_{\mathfrak{N}_1} - (h_j,h_k)_{\mathfrak{N}_2}.$$

We now construct the quadratic form

$$\sum_{j,k=1}^{n}\left(\frac{I-M_2(\lambda_k)^*M_2(\lambda_j)}{1-\lambda_j\overline{\lambda}_k}h_j,h_k\right)_{\mathfrak{N}_2}\xi_j\overline{\xi}_k=\sum_{j,k=1}^{n}[f_{\lambda_j},f_{\lambda_k}]\xi_j\overline{\xi}_k.$$

Since $\mathcal{H}$ has the negative index κ, and since the reduced quadratic form has at most κ negative squares and exactly κ negative squares for some collection f_{λ_j}, we have $M_2(\cdot)\in S_\kappa$. □

Proposition 2.3. *The following relations hold:*

$$-\frac{I-M_1(\mu)^*M_1(\lambda)}{1-\lambda\overline{\mu}}=\gamma_1(\mu)^*\gamma_1(\lambda),\quad \lambda,\mu\in\mathcal{D}_1; \tag{2.20}$$

$$\frac{I-M_2(\mu)^*M_2(\lambda)}{1-\lambda\overline{\mu}}=\gamma_2(\mu)^*\gamma_2(\lambda),\quad \lambda,\mu\in\mathcal{D}_2; \tag{2.21}$$

$$\frac{M_1(\mu)^*-M_2(\lambda)}{1-\lambda\overline{\mu}}=\gamma_1(\mu)^*\gamma_2(\lambda),\quad \lambda\in\mathcal{D}_2,\ \mu\in\mathcal{D}_1; \tag{2.22}$$

$$\frac{M_1(\lambda)-M_2(\mu)^*}{1-\lambda\overline{\mu}}=\gamma_2(\mu)^*\gamma_1(\lambda),\quad \lambda\in\mathcal{D}_1,\ \mu\in\mathcal{D}_2. \tag{2.23}$$

Proof. We now prove (2.20) and (2.22), because (2.21) is proved analogously to (2.20), and (2.23) is a consequence of (2.22).

Let $\lambda,\mu\in\mathcal{D}_1$ and $h_1,h_1'\in\mathfrak{N}_1$. Then formula (2.18) yields

$$\Gamma\begin{bmatrix}\gamma_1(\lambda)h_1\\ \lambda\gamma_1(\lambda)h_1\end{bmatrix}=\begin{bmatrix}h_1\\ M_1(\lambda)h_1\end{bmatrix}\quad\text{and}\quad\Gamma\begin{bmatrix}\gamma_1(\mu)h_1'\\ \mu\gamma_1(\mu)h_1'\end{bmatrix}=\begin{bmatrix}h_1'\\ M_1(\mu)h_1'\end{bmatrix}.$$

Using these identities and setting $\widehat{f}=\widehat{\gamma}_1(\lambda)$ and $\widehat{g}=\widehat{\gamma}_1(\mu)$ in (2.1), we obtain

$$(\lambda\overline{\mu}-1)[\gamma_1(\lambda)h_1,\gamma_1(\mu)h_1']=(h_1,h_1')_{\mathfrak{N}_1}-(M_1(\lambda)h_1,M_1(\mu)h_1')_{\mathfrak{N}_2}$$

or

$$(\lambda\overline{\mu}-1)[\gamma_1(\mu)^*\gamma_1(\lambda)h_1,h_1']=((I-M_1(\mu)^*M_1(\lambda))h_1,h_1')_{\mathfrak{N}_1}.$$

From whence, we obtain equality (2.20).

Let $\lambda\in\mathcal{D}_2$, $\mu\in\mathcal{D}_1$, and let $h_1\in\mathfrak{N}_1$ and $h_2\in\mathfrak{N}_2$. Then formulas (2.18) and (2.19) yield

$$\Gamma\begin{bmatrix}\gamma_2(\lambda)h_2\\ \lambda\gamma_2(\lambda)h_2\end{bmatrix}=\begin{bmatrix}M_2(\lambda)h_2\\ h_2\end{bmatrix}\quad\text{and}\quad\Gamma\begin{bmatrix}\gamma_1(\mu)h_1\\ \mu\gamma_1(\mu)h_1\end{bmatrix}=\begin{bmatrix}h_1\\ M_1(\mu)h_1\end{bmatrix}.$$

From (2.1), we obtain

$$(\lambda\overline{\mu}-1)[\gamma_2(\lambda)h_2,\gamma_1(\mu)h_1]=(M_2(\lambda)h_2,h_1)_{\mathfrak{N}_1}-(h_2,M_1(\mu)h_1)_{\mathfrak{N}_2}.$$

This yields identity (2.22). □

Definition 2.3. *The isometric operator V in $\mathcal{H}$ is called simple, if*

$$\overline{\operatorname{span}}\{\mathfrak{N}_\lambda(V):\lambda\in\hat{\rho}(V)\}=\mathcal{H}.$$

If the isometric operator V in a Pontryagin space $\mathcal{H}$ is simple, then $\mathbb{D}\cup\mathbb{D}_e\in\hat{\rho}(V)$ (see [3]).

Theorem 2.2. *Let* $\Pi = \{\mathfrak{N}_1 \oplus \mathfrak{N}_2, \Gamma_1, \Gamma_2\}$ *be the boundary triplet of a simple isometric operator* V, *and let* $M_1(\cdot)$ *and* $M_2(\cdot)$ *be the functions defined by equalities* (2.16) *and* (2.17). *Then the set of poles of the operator-function* $M_1(\cdot)$ *in* $\mathbb{D}_e$ *coincides with* Λ_1, *and the set of poles of the operator-function* $M_2(\cdot)$ *in* $\mathbb{D}$ *coincides with* Λ_2.

Proof. It follows from (2.20) that if λ_0 is a pole of the operator-function $M_1(\cdot)$, then it is a singular point for $\gamma_1(\cdot)$, i.e., $\lambda_0 \in \Lambda_1$.

Let now $\lambda_0 \in \Lambda_1$. Then

$$(V_1 - \lambda)^{-1} = \frac{A_{-n}}{(\lambda - \lambda_0)^n} + \cdots + \frac{A_{-1}}{\lambda - \lambda_0} + \cdots$$

Let us assume that $M_1(\lambda)$ is holomorphic at the point λ_0. Then the equality

$$-\frac{I - M_1(\mu)^* M_1(\lambda)}{1 - \lambda\overline{\mu}} = \gamma_1(\mu)^* \left(I + (\lambda - \mu')(V_1 - \lambda)^{-1}\right) \gamma_1(\mu')$$

implies that $[A_{-i}\gamma_1(\mu')h_1', \gamma_1(\mu)h_1] = 0$ for all $i = 1, \ldots, n$, $\mu, \mu' \in \mathcal{D}_1$ and any $h_1, h_1' \in \mathfrak{N}_1$.

The equality

$$\frac{M_1(\lambda) - M_2(\mu)^*}{1 - \lambda\overline{\mu}} = \gamma_2(\mu)^* \left(I + (\lambda - \mu')(V_1 - \lambda)^{-1}\right) \gamma_1(\mu')$$

yields $[A_{-i}\gamma_1(\mu')h_1, \gamma_2(\mu)h_2] = 0$ for all $\mu' \in \mathcal{D}_1$, $\mu \in \mathcal{D}_2$ and any $h_1 \in \mathfrak{N}_1$, $h_2 \in \mathfrak{N}_2$. By virtue of the primality of the operator V,

$$\overline{\operatorname{span}}\{\mathfrak{N}_\lambda(V) : \lambda \in \mathcal{D}_1 \cup \mathcal{D}_2\} = \mathcal{H}.$$

Hence, all $A_{-i} = 0$ for $i = 1, \ldots, n$. But this contradicts the assumption that $\lambda_0 \in \Lambda_1$.

The second assertion of this theorem is proved analogously. □

Theorem 2.3. *Under the conditions of the previous theorem, the equality*

$$M_1(\lambda) = M_2(1/\overline{\lambda})^* =: M_2^{\#}(\lambda) \quad \text{for } \lambda \in \mathcal{D}_1 \tag{2.24}$$

holds.

Proof. Let us take $\lambda \in \mathbb{D}_e \backslash (\Lambda_1 \cup \Lambda_2^{\#})$, where $\Lambda_2^{\#}$ is the set symmetric to the set Λ_2 relative to the unit disc. Setting $\mu = 1/\overline{\lambda}$ in (2.23), we obtain (2.24). Equality (2.24) for $\lambda \in \mathcal{D}_1$ can be obtained by the analytic continuation of the function $M_2^{\#}(\lambda) = M_1(\lambda)$ into the points $\lambda \in \mathcal{D}_1 \cap \Lambda_2^{\#}$ and by the application of Theorem 2.2. □

Remark 2.2. By virtue of the holomorphy of $M_1(\cdot)$ in $\mathcal{D}_1$ and $M_2(\cdot)$ in $\mathcal{D}_2$, the identity proved in Theorem 2.3 implies that if λ_0 is a pole of $M_1(\cdot)$, then $1/\overline{\lambda}_0$ is a pole of $M_2(\cdot)$. The same is true for the poles of $M_2(\cdot)$. Thus, the poles of $M_1(\cdot)$ and $M_2(\cdot)$ are symmetric relative to the unit disc. Hence, $\Lambda_1 = \Lambda_2^{\#}$.

Definition 2.4. *The operator-function defined by the equality*

$$M(\lambda) = \begin{cases} M_1(\lambda), & \lambda \in \mathcal{D}_1, \\ M_2(\lambda), & \lambda \in \mathcal{D}_2, \end{cases} \tag{2.25}$$

is called the Weyl function of the operator V, *which corresponds to the boundary triplet* $\Pi = \{\mathfrak{N}_1 \oplus \mathfrak{N}_2, \Gamma_1, \Gamma_2\}$.

Lemma 2.3. *Let $V : \mathcal{H} \to \mathcal{H}$ be an isometric operator, and let the collection $\Pi = \{\mathfrak{N}_1 \oplus \mathfrak{N}_2, \Gamma_1, \Gamma_2\}$ be the boundary triplet of the isometric operator V. Then*

1) *for $\lambda \in \mathcal{D}_1$, the equality*
$$\Gamma_2 \begin{bmatrix} (V_1 - \lambda)^{-1} \\ I + \lambda(V_1 - \lambda)^{-1} \end{bmatrix} = -\frac{1}{\lambda}\gamma_2 \left(\frac{1}{\overline{\lambda}}\right)^* \tag{2.26}$$
holds;

2) *for $\lambda \in \mathcal{D}_2$, the equality*
$$\Gamma_1 \begin{bmatrix} (V_2 - \lambda)^{-1} \\ I + \lambda(V_2 - \lambda)^{-1} \end{bmatrix} = \frac{1}{\lambda}\gamma_1 \left(\frac{1}{\overline{\lambda}}\right)^* \tag{2.27}$$
holds.

Proof. 1) Take $\lambda \in \mathcal{D}_1$, $\mu \in \mathcal{D}_2$, and $h_1 \in \mathfrak{N}_1$. Formula (2.13) yields
$$\widehat{\gamma}_1 (\lambda) h_1 - \widehat{\gamma}_1 (\mu) h_1 = (\lambda - \mu) \begin{bmatrix} (V_1 - \lambda)^{-1} \\ I + \lambda(V_1 - \lambda)^{-1} \end{bmatrix} \gamma_1 (\mu) h_1.$$
Applying the operator Γ_2 to both sides of the equality, we obtain
$$M_1 (\lambda) h_1 - M_1 (\mu) h_1 = (\lambda - \mu)\Gamma_2 \begin{bmatrix} (V_1 - \lambda)^{-1} \\ I + \lambda(V_1 - \lambda)^{-1} \end{bmatrix} \gamma_1 (\mu) h_1.$$
In this formula, we replace $M_1(\lambda)$ by $M_2 \left(\frac{1}{\overline{\lambda}}\right)^*$. In view of formula (2.23), the left-hand side can be written as follows:
$$M_2 \left(\frac{1}{\overline{\lambda}}\right)^* h_1 - M_1(\mu)^* h_1 = -\left(1 - \frac{\mu}{\lambda}\right) \gamma_2 \left(\frac{1}{\overline{\lambda}}\right)^* \gamma_1(\mu)h_1.$$
Equating the right-hand sides of two last formulas, we obtain
$$\Gamma_2 \begin{bmatrix} (V_1 - \lambda)^{-1} \\ I + \lambda(V_1 - \lambda)^{-1} \end{bmatrix} = -\frac{1}{\lambda}\gamma_2 \left(\frac{1}{\overline{\lambda}}\right)^*.$$

2) Take $\lambda \in \mathcal{D}_2$, $\mu \in \mathcal{D}_1$ and $h_2 \in \mathfrak{N}_2$. Substituting λ and μ in formula (2.13), we write it in the form
$$\widehat{\gamma}_2(\lambda)h_2 - \widehat{\gamma}_2(\mu)h_2 = (\lambda - \mu) \begin{bmatrix} (V_2 - \lambda)^{-1} \\ I + \lambda(V_2 - \lambda)^{-1} \end{bmatrix} \gamma_2 (\mu) h_2.$$
Applying the operator Γ_1 to both sides of the equality, we obtain
$$M_2(\lambda)h_2 - M_2(\mu)h_2 = (\lambda - \mu)\Gamma_1 \begin{bmatrix} (V_2 - \lambda)^{-1} \\ I + \lambda(V_2 - \lambda)^{-1} \end{bmatrix} \gamma_2 (\mu) h_2.$$
Replacing $M_2(\lambda)$ in this formula by $M_1 \left(1/\overline{\lambda}\right)^*$, we have
$$M_1 \left(1/\overline{\lambda}\right)^* - M_2(\mu)h_2 = (\lambda - \mu)\Gamma_1 \begin{bmatrix} (V_2 - \lambda)^{-1} \\ I + \lambda(V_2 - \lambda)^{-1} \end{bmatrix} \gamma_2 (\mu) h_2.$$
In view of formula (2.22), we write the left-hand side as
$$M_1 \left(1/\overline{\lambda}\right)^* - M_2(\mu)h_2 = \left(1 - \frac{\mu}{\lambda}\right) \gamma_1 \left(\frac{1}{\overline{\lambda}}\right)^* \gamma_2(\mu)h_2.$$
Comparing the right-hand sides of two last formulas, we obtain formula (2.27). □

2.3. Description of the resolvents of extensions of an isometric operator in a Pontryagin space

Let θ be some closed l.r. from $\mathfrak{N}_2$ in $\mathfrak{N}_1$. Then there exists a Hilbert space H and bounded operators $K_i : H \to \mathfrak{N}_i,\ i = 1, 2$, such that

$$\theta = \left\{ \begin{bmatrix} K_2 h \\ K_1 h \end{bmatrix}, \quad h \in H \right\}. \tag{2.28}$$

Below, we present two theorems, describing the spectrum and the resolvents of extensions V_θ of the operator V. The first theorem gives such a description for the points λ lying outside the unit disc $\mathbb{D}$, i.e., $\lambda \in \mathcal{D}_1 \subset \mathbb{D}_e$.

Theorem 2.4. *Let $V : \mathcal{H} \to \mathcal{H}$ be an isometric operator, let the collection $\Pi = \{\mathfrak{N}_1 \oplus \mathfrak{N}_2, \Gamma_1, \Gamma_2\}$ be the boundary triplet of the isometric operator V, and let θ be the l.r. defined in* (2.28)*. Then, for $\lambda \in \mathcal{D}_1$, the following assertions are valid:*

1) *$\lambda \in \sigma_p(V_\theta)$ iff $0 \in \sigma_p(\theta^{-1} - M_1(\lambda))$. In this case,*

$$\ker(\theta^{-1} - M_1(\lambda)) = \Gamma_1 \begin{bmatrix} f \\ f' \end{bmatrix}, \ \text{where} \ \begin{bmatrix} f \\ f' \end{bmatrix} \in V^{-[*]} \ \text{and} \ f' = \lambda f.$$

2) *$\lambda \in \rho(V_\theta) \cap \mathcal{D}_1$ iff $0 \in \rho(\theta^{-1} - M_1(\lambda))$; for $\lambda \in \rho(V_\theta) \cap \mathcal{D}_1$, the resolvent of the extension V_θ can be determined from the formula*

$$(V_\theta - \lambda)^{-1} = (V_1 - \lambda)^{-1} - \lambda^{-1}\gamma_1(\lambda)\left(\theta^{-1} - M_1(\lambda)\right)^{-1}\gamma_2(\lambda)^{\#}. \tag{2.29}$$

Proof. 1) Let $\lambda \in \sigma_p(V_\theta)$, and let f_λ be an eigenvector V_θ corresponding to the eigenvalue λ. Hence, $\left[\begin{smallmatrix} f_\lambda \\ \lambda f_\lambda \end{smallmatrix}\right] \in V_\theta$, $f_\lambda \in \mathfrak{N}_\lambda(V)$, and $M_1(\lambda)\Gamma_1\widehat{f}_\lambda = \Gamma_2\widehat{f}_\lambda$. Since $f_\lambda \in \operatorname{dom} V_\theta$, we have $\left[\begin{smallmatrix} \Gamma_1\widehat{f}_\lambda \\ \Gamma_2\widehat{f}_\lambda \end{smallmatrix}\right] \in \theta^{-1}$. Hence, $(\theta^{-1} - M_1(\lambda))\Gamma_1\widehat{f}_\lambda = 0$.

Conversely, if $(\theta^{-1} - M_1(\lambda))h_1 = 0$ for some $h_1 \in \mathfrak{N}_1$, then the vector $f_\lambda := \gamma_1(\lambda)h_1 \in \mathfrak{N}_\lambda(V)$, and, hence, $f_\lambda \in \sigma_p(V_\theta)$.

2) Assume that $0 \in \rho(\theta^{-1} - M_1(\lambda))$, $\left[\begin{smallmatrix} f \\ f' \end{smallmatrix}\right] \in V_\theta$ and $g \in \mathcal{H}$. Lemma 2.1 implies that the solution of the equation

$$f' - \lambda f = g \tag{2.30}$$

can be presented in the form

$$\begin{bmatrix} f \\ f' \end{bmatrix} = \begin{bmatrix} f_1 \\ f_1' \end{bmatrix} + \begin{bmatrix} f_\lambda \\ \lambda f_\lambda \end{bmatrix}, \quad \text{where } \widehat{f}_1 \in V_1,\ \widehat{f}_\lambda \in \widehat{\mathfrak{N}}_\lambda(V). \tag{2.31}$$

Then formula (2.30) yields

$$f_1 = (V_1 - \lambda)^{-1} g. \tag{2.32}$$

Applying the operators Γ_1 and Γ_2 to the equality (2.31), we obtain

$$\Gamma_1\widehat{f} = \Gamma_1\widehat{f}_\lambda$$

$$\Gamma_2\widehat{f} = \Gamma_2 \begin{bmatrix} (V_1 - \lambda)^{-1} g \\ g + \lambda(V_1 - \lambda)^{-1} g \end{bmatrix} + \Gamma_2\widehat{f}_\lambda = -\frac{1}{\lambda}\gamma_2\left(\frac{1}{\bar{\lambda}}\right)^{*} g + M_1(\lambda)\Gamma_1\widehat{f}.$$

Since $0 \in \rho(\theta^{-1} - M_1(\lambda))$, the previous equality yields

$$\Gamma_1 \widehat{f}_\lambda = -\frac{1}{\lambda}\left(\theta^{-1} - M_1(\lambda)\right)^{-1} \gamma_2 \left(\frac{1}{\overline{\lambda}}\right)^* g,$$

$$f_\lambda = -\frac{1}{\lambda}\gamma_1(\lambda)\left(\theta^{-1} - M_1(\lambda)\right)^{-1} \gamma_2 \left(\frac{1}{\overline{\lambda}}\right)^* g. \tag{2.33}$$

Equalities (2.31), (2.32), and (2.33) yield equality (2.29).

Conversely, let $\lambda \in \rho(V_\theta)$. By virtue of item 1), to prove the membership $0 \in \rho(\theta^{-1} - M_1(\lambda))$, it is sufficient to show that $\operatorname{ran}(\theta^{-1} - M_1(\lambda)) = \mathfrak{N}_2$. Indeed, by virtue of the surjectivity of the mapping Γ, there exists the vector $\widehat{f}_1 \in V^{-[*]}$ for any $h_2 \in \mathfrak{N}_2$ such that $\Gamma \widehat{f}_1 = \left[\begin{smallmatrix} 0 \\ h_2 \end{smallmatrix}\right]$. Since $\Gamma_1 \widehat{f}_1 = 0$, we have $\widehat{f}_1 \in V_1$. We set $f = (V_\theta - \lambda)^{-1}(f_1' - \lambda f_1)$. Then $f_\lambda := f - f_1 \in \mathfrak{N}_\lambda(V)$ and $f = f_1 + f_\lambda$. Since $\left[\begin{smallmatrix} \Gamma_1 \widehat{f}_\lambda \\ \Gamma_2 \widehat{f}_\lambda \end{smallmatrix}\right] = \left[\begin{smallmatrix} \Gamma_1 \widehat{f} \\ \Gamma_2 \widehat{f}_\lambda \end{smallmatrix}\right] \in \theta^{-1}$, we obtain

$$\Gamma_2 \widehat{f} - M_1(\lambda)\Gamma_1 \widehat{f}_\lambda = \Gamma_2(\widehat{f} - \widehat{f}_\lambda = \Gamma_2 \widehat{f}_1 = h_2.$$

This proves the equality $\operatorname{ran}\left(\theta^{-1} - M_1(\lambda)\right) = \mathfrak{N}_2$ and also the inclusion $0 \in \rho\left(\theta^{-1} - M_1(\lambda)\right)$. □

Corollary 2.1. *If we write the l.r. θ in terms of the operators K_1 and K_2 (see (2.28)), then $\lambda \in \rho(V_\theta)$ iff $0 \in \rho(K_2 - M_1(\lambda)K_1)$. Formula (2.29) takes the form*

$$(V_\theta - \lambda)^{-1} = (V_1 - \lambda)^{-1} - \lambda^{-1}\gamma_1(\lambda)K_1\left(K_2 - M_1(\lambda)K_1\right)^{-1}\gamma_2(\lambda)^{\#}. \tag{2.34}$$

Corollary 2.2. *Let θ be the graph of a unitary operator U from $\mathfrak{N}_2$ to $\mathfrak{N}_1$. Then, for $\lambda \in \mathcal{D}_1$ such that $0 \in \rho(I - M_1(\lambda)U)$, we obtain $\lambda \in \rho(V_\theta)$, and the resolvent of an extension V_θ can be found by the formula*

$$(V_\theta - \lambda)^{-1} = (V_1 - \lambda)^{-1} - \frac{1}{\lambda}\gamma_1(\lambda)U\left(I - M_1(\lambda)U\right)^{-1}\gamma_2\left(\frac{1}{\overline{\lambda}}\right)^*. \tag{2.35}$$

The following result for the points λ inside the unit disc $\mathbb{D}$ can be proved analogously.

Theorem 2.5. *Let $V : \mathcal{H} \to \mathcal{H}$ be an isometric operator, let the collection $\Pi = \{\mathfrak{N}_1 \oplus \mathfrak{N}_2, \Gamma_1, \Gamma_2\}$ be the boundary triplet of an isometric operator V, and let θ be the l.r. defined in (2.28), Then, for $\lambda \in \mathcal{D}_2$, the following assertions are true:*

1) *$\lambda \in \sigma_p(V_\theta)$ iff $0 \in \sigma_p(\theta - M_2(\lambda))$; for $\lambda \in \sigma_p(V_\theta)$, $\ker(\theta - M_2(\lambda)) = \Gamma_2\left[\begin{smallmatrix} f \\ f' \end{smallmatrix}\right]$, where $\left[\begin{smallmatrix} f \\ f' \end{smallmatrix}\right] \in V^{-[*]}$ and $f' = \lambda f$.*

2) *$\lambda \in \rho(V_\theta)$ iff $0 \in \rho(\theta - M_2(\lambda))$; for $\lambda \in \rho(V_\theta) \cap \mathcal{D}_2$, the resolvent of an extension V_θ can be found by the formula*

$$(V_\theta - \lambda)^{-1} = (V_2 - \lambda)^{-1} + \lambda^{-1}\gamma_2(\lambda)\left(\theta - M_2(\lambda)\right)^{-1}\gamma_1(\lambda)^{\#}. \tag{2.36}$$

Corollary 2.3. *If the l.r. θ is written in terms of the operators K_1 and K_2 (see (2.28)), then formula (2.36) takes the form*

$$(V_\theta - \lambda)^{-1} = (V_2 - \lambda)^{-1} + \lambda^{-1}\gamma_2(\lambda)K_2\left(K_1 - M_2(\lambda)K_2\right)^{-1}\gamma_1(\lambda)^{\#}. \tag{2.37}$$

3. Description of the generalized resolvents of an isometric operator in a Pontryagin space

Definition 3.1 ([12]). *The operator-function $\mathbb{R}_\lambda$ holomorphic in a neighborhood $\mathcal{O}$ of the point $\zeta \in \mathcal{D}_1$ is called the generalized resolvent of an isometric operator $V : \mathcal{H} \to \mathcal{H}$, if there exist a Pontryagin space $\widetilde{\mathcal{H}} \supset \mathcal{H}$ and the unitary extension $\widetilde{V} : \widetilde{\mathcal{H}} \to \widetilde{\mathcal{H}}$ of the operator V such that $\zeta \in \rho(\widetilde{V})$, and if the equality*

$$\mathbb{R}_\lambda = P_{\mathcal{H}}\big(\widetilde{V} - \lambda\big)^{-1} \upharpoonright \mathcal{H}, \quad \lambda \in \rho(\widetilde{V}) \cap \mathcal{O} \tag{3.1}$$

in which $P_{\mathcal{H}}$ is the orthoprojector from $\widetilde{\mathcal{H}}$ onto $\mathcal{H}$ holds.

Definition 3.2. *A unitary extension $\widetilde{V}$ of an operator V is called minimal, if $\mathcal{H}_{\widetilde{V}} = \widetilde{\mathcal{H}}$, where*

$$\mathcal{H}_{\widetilde{V}} := \overline{\operatorname{span}}\left\{\mathcal{H} + (\widetilde{V} - \lambda)^{-1}\mathcal{H} : \lambda \in \rho(\widetilde{V})\right\}. \tag{3.2}$$

Proposition 3.1. *Let a unitary extension $\widetilde{V}$ of the operator V be not minimal, $\rho(\widetilde{V}) \neq \emptyset$, and* $\operatorname{ind}_- \widetilde{\mathcal{H}} = \operatorname{ind}_- \mathcal{H} = \kappa$.

Then the following decomposition is valid:

$$\widetilde{\mathcal{H}} = \mathcal{H}_1 \oplus \mathcal{H}_2 \text{ and } \widetilde{V} = \widetilde{V}_1 \oplus \widetilde{V}_2. \tag{3.3}$$

Here, $\mathcal{H}_2 = \mathcal{H}_{\widetilde{V}}$, $\widetilde{V}_2$ is the minimal extension of the operator V, and $\widetilde{V}_1$ is a unitary operator in the Hilbert space $\mathcal{H}_1 \subset \mathcal{H}_{\widetilde{V}}^{\perp}$. In this case,

$$P_{\mathcal{H}}(\widetilde{V} - \lambda)^{-1} \upharpoonright \mathcal{H} = P_{\mathcal{H}}(\widetilde{V}_2 - \lambda)^{-1} \upharpoonright \mathcal{H}. \tag{3.4}$$

Proof. Since $\mathcal{H} \subset \mathcal{H}_{\widetilde{V}} \subset \widetilde{\mathcal{H}}$ and $\operatorname{ind}_- \mathcal{H} = \operatorname{ind}_- \widetilde{\mathcal{H}} = \kappa$, we have $\operatorname{ind}_- \mathcal{H}_{\widetilde{V}} = \kappa$. Hence, $\mathcal{H}_{\widetilde{V}}$ is not degenerate.

We now show that $\mathcal{H}_{\widetilde{V}}$ and $\mathcal{H}_{\widetilde{V}}^{\perp}$ are invariant for $\widetilde{V}$. Let us take different λ_1 and λ_2 from $\rho(\widetilde{V})$. Let $h \in \mathcal{H}$. Then $u := (\widetilde{V} - \lambda_2)^{-1}h \in \mathcal{H}_{\widetilde{V}}$. Let the operator $(\widetilde{V} - \lambda_1)^{-1}$ act on this vector:

$$(\widetilde{V} - \lambda_1)^{-1}(\widetilde{V} - \lambda_2)^{-1}h = \frac{1}{\lambda_1 - \lambda_2}\left((\widetilde{V} - \lambda_1)^{-1} - (\widetilde{V} - \lambda_2)^{-1}\right) h \in \mathcal{H}_{\widetilde{V}}.$$

The case where λ_1 and λ_2 coincide with each other follows from the previous one, if λ_1 tends to λ_2.

Consider now the vectors $v \in \mathcal{H}_{\widetilde{V}}^{\perp}$ and $u \in \mathcal{H}_{\widetilde{V}}$. Then

$$\left[(\widetilde{V} - \lambda)^{-1}v, u\right]_{\widetilde{\mathcal{H}}} = \left[v, (\widetilde{V}^* - \overline{\lambda})^{-1}u\right]_{\widetilde{\mathcal{H}}} = \left[v, \frac{1}{\overline{\lambda}}\left(-I + (I - \overline{\lambda}\widetilde{V})^{-1}\right) u\right]_{\widetilde{\mathcal{H}}} = 0.$$

Here, we use the fact that, for the unitary operator $\widetilde{V}$, the inclusion $\lambda \in \rho(\widetilde{V})$ yields the inclusion $\frac{1}{\overline{\lambda}} \in \rho(\widetilde{V})$.

Thus, $\widetilde{\mathcal{H}} = \mathcal{H}_{\widetilde{V}}^{\perp} \oplus \mathcal{H}_{\widetilde{V}}$ and $\widetilde{V} = \begin{bmatrix} \widetilde{V}_1 & 0 \\ 0 & \widetilde{V}_2 \end{bmatrix}$, where $\widetilde{V}_2$ is the minimal extension of the operator V in $\mathcal{H}_{\widetilde{V}}$.

The equality

$$P_{\mathcal{H}}(I - \lambda\widetilde{V})^{-1} \upharpoonright \mathcal{H} = P_{\mathcal{H}}(I - \lambda\widetilde{V}_2)^{-1} \upharpoonright \mathcal{H}$$

follows from representation (3.3). □

Theorem 3.1. *Let V be an isometry in a Pontryagin space $\mathcal{H}$ with negative index κ, let $\Pi = \{\mathfrak{N}_1 \oplus \mathfrak{N}_2, \Gamma_1, \Gamma_2\}$ be the boundary triplet for V, $V_i = \ker \Gamma_i$, and let $\gamma_i(\cdot)$, $M_i(\cdot)$, $i = 1, 2$, be the corresponding γ-fields and the Weyl functions.*

Let $\widetilde{\mathcal{H}} = \mathcal{H}^{\perp} \oplus \mathcal{H}$ be a Pontryagin space $\operatorname{ind}_{-} \widetilde{\mathcal{H}} = \kappa$. We define the projectors π_1 and π_2 from $\mathcal{H}^{\perp} \times \mathcal{H}^{\perp}$ onto the first and second components in $\mathcal{H}^{\perp} \times \mathcal{H}^{\perp}$,

$$\pi_1 \widehat{h} = h, \quad \pi_2 \widehat{h} = h', \quad \text{where } \widehat{h} = \begin{bmatrix} h \\ h' \end{bmatrix} \in (\mathcal{H}^{\perp})^2.$$

Then

1) *the adjoint l.r. for V^{-1} in the space $\widetilde{\mathcal{H}}$ takes the form*

$$V_{\widetilde{\mathcal{H}}}^{-[*]} = V^{-[*]} \oplus (\mathcal{H}^{\perp})^2; \tag{3.5}$$

2) *the operators*

$$\widetilde{\Gamma}_1 = \begin{bmatrix} \pi_2 & 0 \\ 0 & \Gamma_1 \end{bmatrix} \in [(\mathcal{H}^{\perp})^2 \oplus V^{-[*]}, \mathcal{H}^{\perp} \oplus \mathfrak{N}_1], \tag{3.6}$$

$$\widetilde{\Gamma}_2 = \begin{bmatrix} \pi_1 & 0 \\ 0 & \Gamma_2 \end{bmatrix} \in [(\mathcal{H}^{\perp})^2 \oplus V^{-[*]}, \mathcal{H}^{\perp} \oplus \mathfrak{N}_2] \tag{3.7}$$

are the boundary operators in the boundary triplet $\widetilde{\Pi} = \{(\mathcal{H}^{\perp} \oplus \mathfrak{N}_1) \oplus (\mathcal{H}^{\perp} \oplus \mathfrak{N}_2), \widetilde{\Gamma}_1, \widetilde{\Gamma}_2\}$ for the isometry V in $\widetilde{\mathcal{H}}$.

Moreover,

$$\widetilde{V}_1 (= \ker \widetilde{\Gamma}_1) = V_1 \oplus (\mathcal{H}^{\perp} \oplus \{0\}), \quad \widetilde{V}_2 (= \ker \widetilde{\Gamma}_2) = V_2 \oplus (\{0\} \oplus \mathcal{H}^{\perp}), \tag{3.8}$$

and the corresponding γ-fields and the Weyl functions for the boundary triplet $\widetilde{\Pi}$ take the form

$$\begin{aligned} \widetilde{\gamma}_1(\lambda) &= \begin{bmatrix} \frac{1}{\lambda} I_{\mathcal{H}^{\perp}} & 0 \\ 0 & \gamma_1(\lambda) \end{bmatrix}, \quad \lambda \in \mathcal{D}_1 \\ \widetilde{\gamma}_2(\lambda) &= \begin{bmatrix} I_{\mathcal{H}^{\perp}} & 0 \\ 0 & \gamma_2(\lambda) \end{bmatrix}, \quad \lambda \in \mathcal{D}_2 \end{aligned} \tag{3.9}$$

$$\begin{aligned} \widetilde{M}_1(\lambda) &= \begin{bmatrix} \frac{1}{\lambda} I_{\mathcal{H}^{\perp}} & 0 \\ 0 & M_1(\lambda) \end{bmatrix}, \quad \lambda \in \mathcal{D}_1 \\ \widetilde{M}_2(\lambda) &= \begin{bmatrix} \lambda I_{\mathcal{H}^{\perp}} & 0 \\ 0 & M_2(\lambda) \end{bmatrix}, \quad \lambda \in \mathcal{D}_2 \end{aligned} \tag{3.10}$$

Proof. The first part of the theorem is obvious, and the second one can be verified directly. □

We recall the basic notions of the theory of unitary colligations (see [1,6]). Let $\mathcal{H}$ be a Pontryagin space, let $\mathfrak{N}_2$ and $\mathfrak{N}_1$ be Hilbert spaces, and let $U = \left(\begin{smallmatrix} T & F \\ G & H \end{smallmatrix}\right)$ be a unitary operator from $\mathcal{H} \oplus \mathfrak{N}_2$ to $\mathcal{H} \oplus \mathfrak{N}_1$. Then the quadruple $\Delta = (\mathcal{H}, \mathfrak{N}_2, \mathfrak{N}_1, U)$ is called a unitary colligation. The spaces $\mathcal{H}, \mathfrak{N}_2$, and $\mathfrak{N}_1$ are called, respectively, the space of states, space of inputs, and space of outputs, and the operator U is called the connecting operator of the colligation Δ.

The colligation Δ is called simple, if there exists no subspace in the space $\mathcal{H}$ reducing U. The operator-function

$$\Theta(\lambda) = H + \lambda G (I - \lambda T)^{-1} F : \mathfrak{N}_2 \to \mathfrak{N}_1 \quad (\lambda^{-1} \in \rho(T)) \tag{3.11}$$

is called the characteristic function of a colligation Δ (or the scattering matrix of the unitary operator U relative to the channel spaces $\mathfrak{N}_2$ and $\mathfrak{N}_1$ in the case where $\mathfrak{N}_2, \mathfrak{N}_1, \mathcal{H}$ are Hilbert ones [2]). The characteristic function characterizes a simple unitary colligation to within a unitary equivalence.

Theorem 3.2. *Let V be an isometric operator in $\mathcal{H}$, let $\widetilde{\mathcal{H}} = \mathcal{H} \oplus \mathcal{H}^\perp$ be a Pontryagin space with negative index* $\operatorname{ind}_- \widetilde{\mathcal{H}} = \operatorname{ind}_- \mathcal{H}$, *and let $\widetilde{\Pi}$ be the boundary triplet constructed in Theorem* 3.1.

1) *Any unitary extension $\widetilde{V} \in \mathcal{B}(\widetilde{\mathcal{H}})$ of the operator V can be presented in the form $V = V_\theta := \widetilde{\Gamma}^{-1}\theta$, where θ is the graph of the unitary operator*

$$U = \begin{bmatrix} T & F \\ G & H \end{bmatrix} : \begin{bmatrix} \mathcal{H}^\perp \\ \mathfrak{N}_2 \end{bmatrix} \to \begin{bmatrix} \mathcal{H}^\perp \\ \mathfrak{N}_1 \end{bmatrix}. \tag{3.12}$$

2) *A unitary extension $\widetilde{V} \in \mathcal{B}(\widetilde{\mathcal{H}})$ of the operator V is minimal iff the unitary colligation $\Delta = (\mathcal{H}^\perp, \mathfrak{N}_2, \mathfrak{N}_1; T, F, G, H)$ is simple.*

3) *If $\Theta(\lambda)$ is the characteristic function of the unitary colligation $\Delta = (\mathcal{H}^\perp, \mathfrak{N}_2, \mathfrak{N}_1; T, F, G, H)$, then the generalized resolvent of the operator V, which corresponds to the extension $\widetilde{V}$, takes the following form for $\lambda \in \rho(\widetilde{V}) \cap \mathcal{D}_1$:*

$$\mathbb{R}_\lambda = R_\lambda(V_1) - \frac{1}{\lambda}\gamma_1(\lambda)\Theta\left(\frac{1}{\lambda}\right)\left(I - M_1(\lambda)\Theta\left(\frac{1}{\lambda}\right)\right)^{-1}\gamma_2(\lambda)^\#; \tag{3.13}$$

but if $\lambda \in \rho(\widetilde{V}) \cap \mathcal{D}_2$, it takes the form

$$\mathbb{R}_\lambda = R_\lambda(V_2) + \lambda^{-1}\gamma_2(\lambda)\Theta(\overline{\lambda})^*\left(I - M_2(\lambda)\Theta(\overline{\lambda})^*\right)^{-1}\gamma_1(\lambda)^\#. \tag{3.14}$$

Proof. 1) The assertion of this item of the theorem is a consequence of Theorem 2.1.3.

2) Let the colligation $\Delta = (\mathcal{H}^\perp, \mathfrak{N}_2, \mathfrak{N}_1; T, F, G, H)$ be not simple, i.e., $\mathcal{H}^\perp = \mathcal{H}_1 \oplus \mathcal{H}_2$. Then the unitary operator U takes the form $U = \left[\begin{smallmatrix} U_1 & 0 \\ 0 & U_2 \end{smallmatrix}\right] : \left[\begin{smallmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{smallmatrix}\right]$. In view of operators $\widetilde{\Gamma}_1$ and $\widetilde{\Gamma}_2$ (see formulas (3.6) and (3.7)), we can conclude that they act from $\left[\begin{smallmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{smallmatrix}\right]$ to $\mathcal{H}_1$ as projectors. Hence, $\widetilde{V} = V_\theta$ will have a reducing subspace, namely, $\mathcal{H}_1$. Thus, $\widetilde{V}$ is not the minimal extension of the operator V in $\widetilde{\mathcal{H}}$. The proof of this assertion in the reverse direction is analogous.

3) Using formulas (2.34) and (2.37) for the resolvents of extensions of the operator V, we now find the resolvent of the unitary extension $\widetilde{V} = \widetilde{V}_\theta : \widetilde{\mathcal{H}} \to \widetilde{\mathcal{H}}$, where

$$\theta = \left\{\left\{\begin{bmatrix} h^\perp \\ h \end{bmatrix}, U\begin{bmatrix} h^\perp \\ h \end{bmatrix}\right\},\ \begin{bmatrix} h^\perp \\ h \end{bmatrix} \in \begin{bmatrix} \mathcal{H}^\perp \\ \mathfrak{N}_2 \end{bmatrix}\right\}.$$

Then with regard for (3.1), we obtain

$$\begin{aligned} \mathbb{R}_\lambda g = P_{\mathcal{H}} R_\lambda(\widetilde{V}_1) g - \frac{1}{\lambda} P_{\mathcal{H}} \widetilde{\gamma}_1(\lambda) U \left(I - \widetilde{M}_1(\lambda) U\right)^{-1} \widetilde{\gamma}_2\left(\frac{1}{\overline{\lambda}}\right)^* g \\ = R_\lambda(V_1) g - \frac{1}{\lambda}\gamma_1(\lambda) P_{\mathfrak{N}_1} U \left(I - \widetilde{M}_1(\lambda) U\right)^{-1} \gamma_2\left(\frac{1}{\overline{\lambda}}\right)^* g, \\ \lambda \in \rho(\widetilde{U}_\theta) \cap \mathcal{D}_1,\ g \in \mathcal{H}; \end{aligned} \tag{3.15}$$

$$\mathbb{R}_\lambda g = P_{\mathcal{H}} R_\lambda(\widetilde{V}_2) g + \frac{1}{\lambda} P_{\mathcal{H}} \widetilde{\gamma}_2(\lambda) U^* \left(I - \widetilde{M}_2(\lambda) U^*\right)^{-1} \widetilde{\gamma}_1 \left(\frac{1}{\overline{\lambda}}\right)^* g$$
$$= R_\lambda(V_2) g + \frac{1}{\lambda} \gamma_2(\lambda) P_{\mathfrak{N}_1} U^* \left(I - \widetilde{M}_2(\lambda) U^*\right)^{-1} \gamma_1 \left(\frac{1}{\overline{\lambda}}\right)^* g,$$
$$\lambda \in \rho(\widetilde{U}_\theta) \cap \mathcal{D}_2, \ g \in \mathcal{H}. \quad (3.16)$$

The second formula includes U^*, since

$$\theta = \left\{ \begin{bmatrix} h \\ Uh \end{bmatrix}, \ h \in \begin{bmatrix} \mathfrak{N}_1 \\ \mathcal{H}^\perp \end{bmatrix} \right\} = \left\{ \begin{bmatrix} U^* g \\ g \end{bmatrix}, \ g \in \begin{bmatrix} \mathfrak{N}_2 \\ \mathcal{H}^\perp \end{bmatrix} \right\}$$

by virtue of the unitarity of the operator U.

Using the Frobenius formula for the inverse block matrix, we transform the first formula as

$$\left(I - \widetilde{M}_1(\lambda) U\right)^{-1} = \left(I - \begin{bmatrix} \frac{1}{\lambda} I_{\mathcal{H}^\perp} & 0 \\ 0 & M_1(\lambda) \end{bmatrix} \begin{bmatrix} T & F \\ G & H \end{bmatrix}\right)^{-1}$$
$$= \begin{bmatrix} I - \frac{1}{\lambda} T & -\frac{1}{\lambda} F \\ M_1(\lambda) G & I - M_1(\lambda) H \end{bmatrix}^{-1} = \begin{bmatrix} * & \frac{1}{\lambda}(I - \frac{1}{\lambda} T)^{-1} F \Phi(\frac{1}{\lambda}) \\ * & \Phi(\frac{1}{\lambda}) \end{bmatrix}, \quad (3.17)$$

where $*$ stands for the blocks, which are insignificant, and

$$\Phi\left(\frac{1}{\lambda}\right) := \left(I - M_1(\lambda) H - \frac{1}{\lambda} M_1(\lambda) G \left(I - \frac{1}{\lambda} T\right)^{-1} F\right)^{-1} = \left(I - M_1(\lambda) \Theta\left(\frac{1}{\lambda}\right)\right)^{-1}. \quad (3.18)$$

Substituting (3.17) and (3.18) in (3.15), we obtain

$$P_{\mathfrak{N}_1} U \left(I - \widetilde{M}_1(\lambda) U\right)^{-1} \restriction_{\mathfrak{N}_2} = \frac{1}{\lambda} G (I - \frac{1}{\lambda} T)^{-1} F \Phi(\frac{1}{\lambda}) + H \Phi(\frac{1}{\lambda})$$
$$= \Theta\left(\frac{1}{\lambda}\right) \Phi(\frac{1}{\lambda}) = \Theta\left(\frac{1}{\lambda}\right) \left(I - M_1(\lambda) \Theta\left(\frac{1}{\lambda}\right)\right)^{-1}.$$

The second formula is transformed similarly to the form

$$P_{\mathfrak{N}_2} U^* \left(I - \widetilde{M}_2(\lambda) U^*\right)^{-1} \restriction_{\mathfrak{N}_1} = N(\lambda) \left(I - M_2(\lambda) N(\lambda)\right)^{-1},$$

where

$$N(\lambda) = H^* + \lambda F^* \left(I - \lambda T^*\right)^{-1} G^* = \left(H + \overline{\lambda} G \left(I - \overline{\lambda} T\right)^{-1} F\right)^* = \Theta(\overline{\lambda})^*. \quad (3.19)$$

□

Corollary 3.1. *Under the conditions of Theorem* 3.2, *formulas* (3.13) *and* (3.14) *establish the bijective correspondence between the set of generalized resolvents of the operator V and the set of operator-functions $\Theta \in S_\kappa(\mathfrak{N}_2, \mathfrak{N}_1)$.*

Proof. Let $\mathbb{R}_\lambda$ be the generalized resolvent of the operator V, which admits the representation $\mathbb{R}_\lambda = P_{\mathcal{H}}(\widetilde{V} - \lambda)^{-1} \restriction \mathcal{H}$, where $\widetilde{V}$ is the minimal extension of the operator V in the space $\widetilde{\mathcal{H}}$. Then, by virtue of Theorem 3.2, the operator $\widetilde{V}$ admits the representation $\widetilde{V} = \widetilde{\Gamma}^{-1}\theta$, where θ is the graph of the unitary operator U from (3.12). In this case, the unitary colligation is simple. The characteristic function $\Theta(\cdot)$ of this colligation belongs to the class $S_\kappa(\mathfrak{N}_2, \mathfrak{N}_1)$. By virtue of Theorem 3.2.3, formulas (3.13) and (3.14) are valid.

Conversely, if $\Theta(\cdot) \in S_\kappa(\mathfrak{N}_2, \mathfrak{N}_1)$, then there exists a simple unitary colligation Δ that is uniquely determined to within a unitary equivalence and is such that the characteristic function of this colligation coincides with $\Theta(\cdot)$. Let us consider the boundary triplet from Theorem 3.1. The extension $\widetilde{V}$ is minimal. By virtue of Theorem 3.2.3, the generalized resolvent corresponding to $\widetilde{V}$, can be found by formulas (3.13) and (3.14). □

REFERENCES

1. D. Alpay, A. Dijksma, J. Rovnyak, and H. S. V. de Snoo, *Schur Functions, Operator Colligations, and Reproducing Kernel Pontryagin Spaces*, Birkhäuser, Basel, 1997.
2. D. Z. Arov and L. Z. Grossman, "The scattering matrices in the theory of the extensions of isometric operators," *DAN SSSR*, **33**, No. 4, 17–20 (1983).
3. T. Ya. Azizov and I. S. Iokhvidov, *Foundations of the Theory of Operators in Spaces with Indefinite Metric* [in Russian], Nauka, Moscow, 1986.
4. D. V. Baidiuk, "On the scattering matrices of the unitary extensions of isometric operators in the Pontryagin space," *Trudy KROMSh*, **19**, 1–6 (2009).
5. C. Benewitz, "Symmetric relations on a Hilbert space," *Lect. Notes Math.*, **280**, 212–218 (1972).
6. M. S. Brodskii, "Unitary operator colligations and their characteristic functions," *Uspekhi Mat. Nauk*, **33**, No. 4, 141–168 (1978).
7. V. A. Derkach, S. Hassi, M. M. Malamud, and H. S. V. de Snoo, "Boundary relations and their Weyl families," *Trans. of AMS*, **358**, No. 12, 5351–5400 (2006).
8. A. Dijksma, H. Langer, and H. S. V. de Snoo, "Generalized coresolvents of standard isometric relations and generalized resolvent of standard symmetric relations in Kreĭn space," in *Operator Theory: Advances and Applicatios*, Birkhäuser, Basel, 1990, pp. 261–274.
9. I. S. Iokhvidov and M. G. Krein, "The spectral theory of operators in spaces with indefinite metric, I," *Trudy Mosk. Mat. Obshch.*, **5**, 367–432 (1965).
10. E. I. Iokhvidov, "The description of the J-unitary extensions of J-isometric operators with equal and finite defect numbers," *DAN SSSR*, **228**, No. 4, 783–786 (1976).
11. V. E. Katsnel'son, A. Ya. Kheifets, and P. M. Yuditskii, "An abstract interpolational problem and the theory of the extensions of isometric operators," in *Operators in Functional Spaces and Questions of the Theory of Functions* [in Russian], Naukova Dumka, Kiev, 1987, pp. 83–96.
12. G. K. Langer, "The generalized coresolvents of a Π-isometric operator with unequal defect numbers," *Funkts. Anal. Priloz.*, **5**, No. 4, 73–75 (1971).
13. M. M. Malamud and V. I. Mogilevskii, "The generalized resolvents of an isometric operator," *Mat. Zametki*, **73**, No. 3, 460–466 (2003).
14. M. M. Malamud and V. I. Mogilevskii, "The resolvent matrices and the spectral functions of an isometric operator," *Dokl. Akad. Nauk*, **395**, No. 1, 11–17 (2004).
15. O. Nitz, "Generalized resolvents of isometric linear relations in Pontryagin space, 1: Foundations," in *Operator Theory: Advances and Applications*, Birkhäuser, Basel, 2000, pp. 303–319.
16. O. Nitz, "Generalized resolvents of isometric linear relations in Pontryagin space, 2: Krein–Langer formula," *Meth. Funct. Anal. Topol.*, **6**, No. 3, 72–96 (2000).

17. Yu. L. Shmul'yan, "Theory of linear relations and spaces with indefinite metric," *Funkts. Anal. Pril.*, **10**, No. 1, 67–72 (1976).

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SHORT COMMUNICATIONS

Description of Scattering Matrices of Unitary Extensions of Isometric Operators in Pontryagin Space

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1. INTRODUCTION

Many problems in analysis, such as the Nevanlinna–Pick problem, the moment problem, etc., can be reduced to the study of some symmetric operator A in a Hilbert space H. Here, an important role for describing $\mathfrak{L}$-resolvents is played by Krein's formula [1] for a symmetric operator A with finite deficiency indices and extended in [2] to the case of operators with infinite deficiency indices. An explicit formula for the $\mathfrak{L}$-resolvent matrix of a symmetric operator A in terms of boundary triplets was established in [3] (for the case of symmetric operators in Pontryagin space, see [4]).

In the case of an isometric operator V in Hilbert space, the analog of Krein's formula for $\mathfrak{L}$-resolvents is the formula describing the scattering matrices of unitary extensions of an operator V; it was derived by Arov and Grossman in [5]. The Arov–Grossman formula served as a basis for constructing a theory of abstract interpolation in the papers of Katsnel'son, Kheifets, and Yuditskii (see [6]), including the majority of the well-known classical interpolation problems.

In the papers of Malamud and Mogilevskii [7], [8], the boundary operator method was extended to the case of isometric operators and applied to the problem of describing the generalized resolvents of an isometric operator V in Hilbert space. In [9], this method was further developed to the case of isometric operators acting in the Pontryagin spaces Π_κ, and it served as a basis for describing the generalized resolvents of isometric operators in the spaces Π_κ.

In the present paper, we use these results to describe the scattering matrices of isometric operators V in Π_κ and obtain explicit formula for the $\mathfrak{L}$-resolvent matrices of the operator V. In what follows, this formula will be applied to the description of solutions of an indefinite abstract interpolation problem and, in particular, to such problems as the Nevanlinna–Pick problem and the bitangent interpolation problem for the generalized Schur classes.

2. BOUNDARY TRIPLETS AND THE WEYL FUNCTION

Let $(\mathcal{H}, [\cdot\,,\,\cdot])$ be a Pontryagin space with negative index $\kappa = \operatorname{ind}_- \mathcal{H}$ (see [10]), let $\mathfrak{N}_1$, $\mathfrak{N}_2$, and $\mathfrak{L}_1 \subseteq \mathfrak{L}_2$ be Hilbert spaces, and let $\mathcal{B}(\mathfrak{N}_2, \mathfrak{N}_1)$ be the set of bounded linear operators from $\mathfrak{N}_2$ to $\mathfrak{N}_1$. Let V be an isometric operator from $\mathcal{H} \oplus \mathfrak{L}_2$ to $\mathcal{H} \oplus \mathfrak{L}_1$, and let $V^{[*]}$ be the adjoint linear relation from $\mathcal{H} \oplus \mathfrak{L}_1$ to $\mathcal{H} \oplus \mathfrak{L}_2$. For brevity, we shall write $V^{-[*]} := (V^{[*]})^{-1}$. Note that

$$\operatorname{gr} V = \left\{ \begin{bmatrix} f \\ Vf \end{bmatrix} : f \in \operatorname{dom} V \right\} \subset V^{-[*]}.$$

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Denote

$$\mathbb{D} = \{\lambda \in \mathbb{C} : \|\lambda\| < 1\}, \qquad \mathbb{D}_e = \{\lambda \in \mathbb{C} : \|\lambda\| > 1\}, \qquad \mathbb{T} = \{\lambda : \|\lambda\| = 1\},$$

and $S(\mathfrak{N}_2, \mathfrak{N}_1)$ is the Schur class of contraction, analytic (in $\mathbb{D}$) operator functions with values in $\mathcal{B}(\mathfrak{N}_2, \mathfrak{N}_1)$

Definition 1. The set $\Pi = \{\mathfrak{N}_1 \oplus \mathfrak{N}_2, \Gamma_1, \Gamma_2\}$ will be called the *boundary triplet* of an isometric operator V if

1) the mapping $\Gamma = (\Gamma_1, \Gamma_2)^T : V^{-[*]} \to \mathfrak{N}_1 \oplus \mathfrak{N}_2$ is surjective;

2) the generalized Green identity

$$[f', g']_{\mathcal{H}\oplus\mathfrak{L}_1} - [f, g]_{\mathcal{H}\oplus\mathfrak{L}_2} = (\Gamma_1 \widehat{f}, \Gamma_1 \widehat{g})_{\mathfrak{N}_1} - (\Gamma_2 \widehat{f}, \Gamma_2 \widehat{g})_{\mathfrak{N}_2} \tag{1}$$

holds for all

$$\widehat{f} = \begin{bmatrix} f \\ f' \end{bmatrix}, \qquad \widehat{g} = \begin{bmatrix} g \\ g' \end{bmatrix} \in V^{-[*]}.$$

Remark 1. If $\{\mathfrak{N}_1 \oplus \mathfrak{N}_2, \Gamma_1, \Gamma_2\}$ is a boundary triplet for the isometry V and $\dim \mathfrak{N}_1 = \dim \mathfrak{L}_1$, $\dim \mathfrak{N}_2 = \dim \mathfrak{L}_2$, then there exist unitary operators $X_i : \mathfrak{N}_i \to \mathfrak{L}_i$, $i = 1, 2$ such that the set

$$\{\mathfrak{L}_1 \oplus \mathfrak{L}_2, X_1\Gamma_1, X_2\Gamma_2\}$$

is a boundary triplet for V. In this case, a boundary triplet for V can be chosen so that $\mathfrak{N}_1 = \mathfrak{L}_1$ and $\mathfrak{N}_2 = \mathfrak{L}_2$.

Proposition 1. *If $\iota : \mathcal{H} \oplus \mathfrak{L}_1 \hookrightarrow \mathcal{H} \oplus \mathfrak{L}_2$ is an embedding operator and $\widetilde{V} f := \iota V f$ is an isometry from $\mathcal{H} \oplus \mathfrak{L}_2$ to $\mathcal{H} \oplus \mathfrak{L}_2$, then*

$$\widetilde{V}^{-[*]} = V^{-[*]} \dotplus \left\{ \begin{bmatrix} 0 \\ u \end{bmatrix} : u \in \mathfrak{L}_2 \ominus \mathfrak{L}_1 \right\}.$$

Proposition 2. *If $\{\mathfrak{N}_1 \oplus \mathfrak{N}_2, \Gamma_1, \Gamma_2\}$ is a boundary triplet for $V : \mathcal{H} \oplus \mathfrak{L}_2 \to \mathcal{H} \oplus \mathfrak{L}_1$, then the set $\{\widetilde{\mathfrak{N}}_1 \oplus \mathfrak{N}_2, \widetilde{\Gamma}_1, \widetilde{\Gamma}_2\}$, where $\widetilde{\mathfrak{N}}_1 = \mathfrak{N}_1 \oplus (\mathfrak{L}_2 \ominus \mathfrak{L}_1)$,*

$$\widetilde{\Gamma}_1\left(\widehat{f} + \begin{bmatrix} 0 \\ u \end{bmatrix}\right) = \begin{bmatrix} \Gamma_1 \widehat{f} \\ u \end{bmatrix}, \quad \widetilde{\Gamma}_2\left(\widehat{f} + \begin{bmatrix} 0 \\ u \end{bmatrix}\right) = \Gamma_2 \widehat{f}, \qquad \widehat{f} \in \widetilde{V}^{-[*]}, \quad u \in \mathfrak{L}_2 \ominus \mathfrak{L}_1,$$

constitutes a boundary triplet for $\widetilde{V} : \mathcal{H} \oplus \mathfrak{L}_2 \to \mathcal{H} \oplus \mathfrak{L}_2$.

For an isometric operator $\widetilde{V}$, the defect subspaces $\mathfrak{N}_\lambda(\widetilde{V})$ are defined as follows:

$$\mathfrak{N}_\lambda(\widetilde{V}) := \left\{ f_\lambda : \begin{bmatrix} f_\lambda \\ \lambda f_\lambda \end{bmatrix} \in \widetilde{V}^{-[*]} \right\}. \tag{2}$$

In the case of a Hilbert space $\mathcal{H}$, the notion of boundary triplet of an isometric operator V was introduced in [7] and, in the case of a Pontryagin space, in [11].

3. REPRESENTATION THEORY OF ISOMETRIC OPERATORS

Let $P_{\mathcal{H}}$ and $P_{\mathfrak{L}_1}$ be the orthogonal projections onto $\mathcal{H}$ and $\mathfrak{L}_1$, and let $\widehat{\rho}(P_{\mathcal{H}}V)$ be the set of points of regular type of the operator $P_{\mathcal{H}}V$ (see [13]).

Definition 2. We shall write $\lambda \in \rho(V, \mathfrak{L}_2)$ if $1 \in \widehat{\rho}(\lambda P_{\mathcal{H}}V)$ and

$$(I - \lambda P_{\mathcal{H}}V)\operatorname{dom} V \dotplus \mathfrak{L}_2 = \mathcal{H} \oplus \mathfrak{L}_2, \tag{3}$$

and $\lambda \in \rho(V^{-1}, \mathfrak{L}_2)$ if $1 \in \widehat{\rho}(\lambda P_{\mathcal{H}}V^{-1})$ and

$$(I - \lambda P_{\mathcal{H}}V^{-1})\operatorname{ran} V \dotplus \mathfrak{L}_1 = \mathcal{H} \oplus \mathfrak{L}_1. \tag{4}$$

We shall write $\lambda \in \rho_V(\mathfrak{L}_2, \mathfrak{L}_1)$ if $\lambda \in \rho(V, \mathfrak{L}_2)$, and $\overline{\lambda} \in \rho(V^{-1}, \mathfrak{L}_1)$.

Definition 3. For $\lambda \in \rho(V, \mathfrak{L}_2)$, by $\mathcal{P}_{\mathfrak{L}_2}(\lambda)$ we denote the skew projection onto $\mathfrak{L}_2$ in the decomposition (3) and introduce the operators

$$\mathcal{Q}_{\mathfrak{L}_2}(\lambda) := P_{\mathfrak{L}_2}\widetilde{V}(I - \lambda P_{\mathcal{H}}V)^{-1}(I - \mathcal{P}_{\mathfrak{L}_2}(\lambda)) : \mathcal{H} \oplus \mathfrak{L}_2 \to \mathfrak{L}_2,$$
$$\mathcal{Q}_{\mathfrak{L}_1}(\lambda) := P_{\mathfrak{L}_1}\mathcal{Q}_{\mathfrak{L}_2}(\lambda).$$

We also define

$$\widehat{\mathcal{P}}_{\mathfrak{L}_2}(\lambda)^{[*]} = \begin{bmatrix} \mathcal{P}_{\mathfrak{L}_2}(\lambda)^{[*]} \\ \overline{\lambda}(\mathcal{P}_{\mathfrak{L}_2}(\lambda)^{[*]} - I_{\mathfrak{L}_2}) \end{bmatrix}, \quad \widehat{\mathcal{Q}}_{\mathfrak{L}_i}(\lambda)^{[*]} = \begin{bmatrix} \mathcal{Q}_{\mathfrak{L}_i}(\lambda)^{[*]} \\ \overline{\lambda}\mathcal{Q}_{\mathfrak{L}_i}(\lambda)^{[*]} + I_{\mathfrak{L}_i} \end{bmatrix}, \quad \text{for} \quad i = 1, 2. \tag{5}$$

Further,

$$\mathfrak{N}_{\overline{\lambda}}(\widetilde{V}) = (\mathcal{P}_{\mathfrak{L}_2}(\lambda)^{[*]} + \overline{\lambda}\mathcal{Q}_{\mathfrak{L}_2}(\lambda)^{[*]})\mathfrak{L}_2.$$

Theorem 1. *For $\lambda \in \rho_V(\mathfrak{L}_2, \mathfrak{L}_1)$, the following decompositions are valid:*

$$V^{-[*]} = \operatorname{gr} V \dotplus \widehat{\mathcal{P}}_{\mathfrak{L}_2}(\lambda)^{[*]}\mathfrak{L}_2 \dotplus \mathcal{Q}_{\mathfrak{L}_1}(\lambda)^{[*]}\mathfrak{L}_1. \tag{6}$$
$$\widetilde{V}^{-[*]} = \operatorname{gr} \widetilde{V} \dotplus \widehat{\mathcal{P}}_{\mathfrak{L}_2}(\lambda)^{[*]}\mathfrak{L}_2 \dotplus \mathcal{Q}_{\mathfrak{L}_2}(\lambda)^{[*]}\mathfrak{L}_2. \tag{7}$$

For $\lambda \in \rho(V, \mathfrak{L}_2)$, we define the operators

$$\mathcal{G}(\lambda) := \begin{bmatrix} \mathcal{Q}_{\mathfrak{L}_1}(\lambda) \\ I_{\mathfrak{L}_2} - \mathcal{P}_{\mathfrak{L}_2}(\lambda) \end{bmatrix}, \qquad \widetilde{\mathcal{G}}(\lambda) := \begin{bmatrix} \mathcal{Q}_{\mathfrak{L}_2}(\lambda) \\ I_{\mathfrak{L}_2} - \mathcal{P}_{\mathfrak{L}_2}(\lambda) \end{bmatrix}, \tag{8}$$

$$\mathcal{V}(\lambda) := \begin{bmatrix} \widehat{\mathcal{Q}}_{\mathfrak{L}_1}(\lambda) \\ -\widehat{\mathcal{P}}_{\mathfrak{L}_2}(\lambda) \end{bmatrix}, \qquad \widetilde{\mathcal{V}}(\lambda) := \begin{bmatrix} \widehat{\mathcal{Q}}_{\mathfrak{L}_2}(\lambda) \\ -\widehat{\mathcal{P}}_{\mathfrak{L}_2}(\lambda) \end{bmatrix}. \tag{9}$$

Theorem 2. *Let $\Pi = \{\mathfrak{N}_1 \oplus \mathfrak{N}_2, \Gamma_1, \Gamma_2\}$ be a boundary triplet for $V: \mathcal{H} \oplus \mathfrak{L}_2 \to \mathcal{H} \oplus \mathfrak{L}_1$, and let $\widetilde{\Pi} = \{\widetilde{\mathfrak{N}}_1 \oplus \mathfrak{N}_2, \widetilde{\Gamma}_1, \widetilde{\Gamma}_2\}$ be a boundary triplet for $\widetilde{V} = \iota V: \mathcal{H} \oplus \mathfrak{L}_2 \to \mathcal{H} \oplus \mathfrak{L}_2$. Then, for $\lambda, \mu \in \rho_V(\mathfrak{L}_2, \mathfrak{L}_1)$, the operator functions*

$$W(\lambda) := (\Gamma\mathcal{V}(\lambda)^{[*]})^*, \qquad \widetilde{W}(\lambda) := (\widetilde{\Gamma}\widetilde{\mathcal{V}}(\lambda)^{[*]})^* \tag{10}$$

satisfy the identities

$$J_{\mathfrak{L}} - W(\lambda)J_{\mathfrak{N}}W(\mu)^* = (1 - \lambda\overline{\mu})\mathcal{G}(\lambda)\mathcal{G}(\mu)^{[*]}, \tag{11}$$
$$J_{\widetilde{\mathfrak{L}}} - \widetilde{W}(\lambda)J_{\widetilde{\mathfrak{N}}}\widetilde{W}(\mu)^* = (1 - \lambda\overline{\mu})\widetilde{\mathcal{G}}(\lambda)\widetilde{\mathcal{G}}(\mu)^{[*]}, \tag{12}$$

where

$$J_{\mathfrak{L}} = \begin{bmatrix} I_{\mathfrak{L}_1} & 0 \\ 0 & -I_{\mathfrak{L}_2} \end{bmatrix}, \qquad J_{\mathfrak{N}} = \begin{bmatrix} I_{\mathfrak{N}_1} & 0 \\ 0 & -I_{\mathfrak{N}_2} \end{bmatrix},$$

$$J_{\widetilde{\mathfrak{L}}} = \begin{bmatrix} I_{\mathfrak{L}_2} & 0 \\ 0 & -I_{\mathfrak{L}_2} \end{bmatrix}, \qquad J_{\widetilde{\mathfrak{N}}} = \begin{bmatrix} I_{\mathfrak{N}_2} & 0 \\ 0 & -I_{\mathfrak{N}_2} \end{bmatrix}.$$

Definition 4. The operator functions $W(\cdot)$ and $\widetilde{W}(\cdot)$ satisfying identities (11) and (12) are called the *resolvent matrices* of the operators V and $\widetilde{V}$, respectively.

Thus, $W(\cdot)$ and $\widetilde{W}(\cdot)$ defined by formulas (10) are the resolvent matrices for the operators V and $\widetilde{V}$, respectively. Further,

$$W(\cdot) = P_{\mathfrak{L}_1 \oplus \mathfrak{L}_2} \widetilde{W}(\cdot) \upharpoonright \mathfrak{N}_1 \oplus \mathfrak{N}_2.$$

Remark 2. If $\Pi = \{\mathfrak{N}_1 \oplus \mathfrak{N}_2, \Gamma_1, \Gamma_2\}$ is a boundary triplet such that $\mathfrak{N}_1 = \mathfrak{L}_1$, $\mathfrak{N}_2 = \mathfrak{L}_2$, and $a \in \mathbb{T} \cap \rho_V(\mathfrak{L}_2, \mathfrak{L}_1)$ $(\neq \varnothing)$, then, setting $W(a) = I$ and using (11), we obtain

$$W(\lambda) = I - (1 - \lambda\overline{a})\mathcal{G}(\lambda)\mathcal{G}(a)^{[*]} J_{\mathfrak{L}}, \qquad \lambda \in \rho_V(\mathfrak{L}_2, \mathfrak{L}_1). \tag{13}$$

Definition 5. Let $U \colon \widetilde{\mathcal{H}} \oplus \mathfrak{L}_2 \to \widetilde{\mathcal{H}} \oplus \mathfrak{L}_1$ be the unitary operator which is the extension of an isometric operator $V \colon \mathcal{H} \oplus \mathfrak{L}_2 \to \mathcal{H} \oplus \mathfrak{L}_1$, and let $\widetilde{\mathcal{H}}$ be a Pontryagin space containing $\mathcal{H}$ such that $\operatorname{ind}_- \widetilde{\mathcal{H}} = \operatorname{ind}_- \mathcal{H}$. The operator function $s(\lambda) \colon \mathfrak{L}_2 \to \mathfrak{L}_1$ defined in $\mathbb{D} \setminus (\sigma_p(U) \cup \sigma_p(UP_{\mathcal{H}}))^{-1}$ by the equality

$$s(\lambda) = P_{\mathfrak{L}_1}(I - \lambda U P_{\mathcal{H}})^{-1} U \upharpoonright \mathfrak{L}_2, \tag{14}$$

is called the *scattering matrix* of the unitary extension U of an isometric operator V (see [5], [11], [9]).

Theorem 3. *Let* $0 \in \rho_V(\mathfrak{L}_2, \mathfrak{L}_1)$*, and let*

$$W(\lambda) = \begin{bmatrix} w_{11}(\lambda) & w_{12}(\lambda) \\ w_{21}(\lambda) & w_{22}(\lambda) \end{bmatrix}$$

be the matrix function defined by relation (10)*. Then the set of scattering matrices of various unitary extensions of the operator V is described by the formula*

$$s(\lambda) = (w_{12}(\lambda) + w_{11}(\lambda)\varepsilon(\lambda))(w_{22}(\lambda) + w_{21}(\lambda)\varepsilon(\lambda))^{-1}, \qquad \lambda \in \rho_V(\mathfrak{L}_2, \mathfrak{L}_1), \tag{15}$$

where the parameter $\varepsilon(\cdot)$ *belongs to the Schur class* $S(\mathfrak{N}_2, \mathfrak{N}_1)$ *and satisfies the condition* $0 \in \rho(w_{22}(0) + w_{21}(0)\varepsilon(0))$.

Formula (15) follows from the similar scattering matrix $\widetilde{s}(\lambda)$ for $\widetilde{V}$:

$$\widetilde{s}(\lambda) = (\widetilde{w}_{12}(\lambda) + \widetilde{w}_{11}(\lambda)\widetilde{\varepsilon}(\lambda))(\widetilde{w}_{22}(\lambda) + \widetilde{w}_{21}(\lambda)\widetilde{\varepsilon}(\lambda))^{-1}, \qquad \lambda \in \rho_V(\mathfrak{L}_2, \mathfrak{L}_1), \tag{16}$$

where the parameter $\widetilde{\varepsilon}(\cdot)$ ranges over the set $S(\mathfrak{N}_2, \mathfrak{N}_2)$ and the $\widetilde{w}_{ij}(\cdot)$ are the elements of the resolvent matrix $\widetilde{W}(\cdot)$ for $\widetilde{V}$. In proving formula (16), the formula for the generalized resolvents of the isometry $\widetilde{V}$ proved in [9] is used. Namely, it is this fact that makes it necessary to consider the extended operator $\widetilde{V}$, for which the defect subspaces and spectrum is well defined, in contrast to the operator V.

Remark 3. In the case of a Hilbert space, formula (15) was obtained in [5]. In [8], this formula, as well as Theorems 1–3, was obtained by the boundary operator method. In the indefinite case, the representation theory of standard isometric operators (i.e., with nondegenerate $\operatorname{dom} V$) was studied in [13], [14]. Note that, in our approach, first, we are able to construct the representation theory of nonstandard isometric operators in Π_κ and, second, simplify some results from [13], in particular, formula (11) by using another definition of the operator functions $\mathcal{P}_{\mathfrak{L}_2}(\lambda)$ and $\mathcal{Q}_{\mathfrak{L}_2}(\lambda)$. For symmetric operators, the corresponding results (Theorems 1–3) were obtained in [3] and [5].

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REFERENCES

1. M. G. Krein, Dokl. Akad. Nauk SSSR **52**, 651 (1946).
2. M. G. Krein and Sh. N. Saakyan, Dokl. Akad. Nauk SSSR **169**, 1269 (1966).
3. V. A. Derkach and M. M. Malamud, J. Funct. Anal. **95** (1), 1 (1991).
4. V. A. Derkach, J. Math. Sci. (New York) **97** (5), 4420 (1999).
5. D. Z. Arov and L. Z. Grossman, Dokl. Ross. Akad. Nauk [Dokl. Math.] Dokl. Akad. Nauk SSSR **270** (1), 17 (1983) [Soviet Math. Dokl. **27** (3), 518 (1983)].
6. M. M. Malamud and V. I. Mogilevskii, Mat. Zametki **73** (3), 460 (2003) [Math. Notes **73** (3), 429 (2003)].
7. M. M. Malamud and V. I. Mogilevskii, Dokl. Ross. Akad. Nauk **395** (1), 11 (2004).
8. V. E. Katsnelson, A. Ya. Kheifets, and P. M. Yuditskii, in *Topics in Interpolation Theory*, *Oper. Theory Adv. Appl.* (Birkhäuser Verlag, Basel, 1997), Vol. 95, pp. 283–298 [in Russian].
9. T. Ya. Azizov and I. S. Iokhvidov, *Foundations of the Theory of Linear Operators in Spaces with Indefinite Metric* (Nauka, Moscow, 1986) [in Russian].
10. D. V. Baidiuk, Ukrain. Mat. Vestnik **10** (2), 176 (2013).
11. D. V. Baidiuk, Trudy Krymsk. Osennei Mat. Shkoly **19**, 1 (2009).
12. N. I. Akhiezer and I. M. Glazman, *Theory of Linear Operators in Hilbert Space* (Vyshcha Shkola, Kharkov, 1977), Vol. 1 [in Russian].
13. H. Langer, Funktsional. Anal. Prilozhen. **5** (4), 73 (1971) [Functional Anal. Appl. **5**, 329 (1972)].
14. H. Langer and P. Sorjonen, Ann. Acad. Sci. Fenn. Ser. A I **561**, 3 (1974).